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Interpolation of p-Adic Holomorphic Functions

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Abstract. We give a necessary and sufficient condition for interpolation sequences of p-adic holomorphic functions on $\mathbb{C}_p^*$.

1. Introduction

With the construction of the p-adic zeta - function by interpolating a set of integers many more people are interested in the problem of p-adic interpolation. The p-adic interpolation is considered by Mahler in [8], and is investigated systematically by Amice [2], but only for bounded functions. In [4] Khoai proved a p-adic interpolation theorem for holomorphic functions by interpolating a discrete sequence of points in the unit disc. This is the first theorem on p-adic interpolations of unbounded functions. In [6] Khoai introduced the notion of height for a p-adic holomorphic function and for a sequence of points, and formulated and proved an interpolation theorem giving the necessary and sufficient condition for a discrete sequence of points to be an interpolating sequence of a given function. Note that in [6] the author considered only functions on the unit disc.

In [5] Khoai and in [7] Khoai and Quang applied the p-adic interpolation to the construction of an analog of Nevanlinna theory for p-adic holomorphic functions on the unit disc. In [3] Cherry obtained p-adic analog of Nevanlinna theory on the puntured plane $\mathbb{C}_p^*$. In this paper, motivated by the above mentioned results we consider the problem of p-adic interpolation on $\mathbb{C}_p^*$, and as a consequence, obtain an interpolation theorem for holomorphic functions on $\mathbb{C}_p$.

To formulate and prove the interpolation theorem we introduce the notion of height for a p-adic holomorphic function on $\mathbb{C}_p^*$ and give an analog of the Poisson–Jensen formula.

2. The Height of a p-Adic Holomorphic Function and the Height of a Sequense of Points on $\mathbb{C}_p^*$

In this section we give the notion of height of a p-adic holomorphic function, height of a sequence of points in $\mathbb{C}_p^*$, and construct the Poisson-Jensen formula of p-adic holomorphic functions on $\mathbb{C}_p^*$.

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers and $\mathbb{C}_p$ the completion of the algebraic closure of $\mathbb{Q}_p$. The absolute value in $\mathbb{C}_p$ is normalized so that $|p| = p^{-1}$. We further use $v(z)$ to denote the additive valuation on $\mathbb{C}_p$, which extends ord_p.

For $\mu, \nu \in \mathbb{R} \cup \{-\infty, +\infty\}$, let

$$C(\mu,\nu) = \{z \in \mathbb{C}_p \mid \mu < v(z) < \nu\},$$
$$C[\mu,\nu] = \{z \in \mathbb{C}_p \mid \mu \leq v(z) \leq \nu\},$$
$$\mathbb{C}_p^* = C(-\infty,\infty) \text{ denote the puntured } p\text{-adic plane.}$$

Let $f(z)$ be a p-adic holomorphic function on $\mathbb{C}_p^*$, defined by a Laurent series:

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n.$$

This means that, for all $t \in \mathbb{R}$

$$\lim_{|n|\to\infty} (v(a_n) + nt) = +\infty, \quad t = v(z).$$

Therefore for each $t \in \mathbb{R}$ there exists an integer n such that $v(a_n)+nt$ attains minimum.

Definition 1. *The height of a p-adic holomorphic function $f(z)$ is defined by*

$$H(f,t) = \min_{n\in\mathbb{Z}}\{v(a_n) + nt\}.$$

Now we give a geometric interpretation of the notion of height.

For each n we draw the graph Γ_n which depicts $v(a_n z^n) = v(a_n) + nv(z)$ as a function of $v(z) = t$. This graph is a straight line with slope n. From the definition of $H(f,t)$ we see that $H(f,t)$ is the boundary of all of half- planes lying under the lines Γ_n. Thus, the graph $H(f,t)$ is a polygon line, which is called the Newton polygon of $f(z)$.

Let

$$n_{f,t}^+ = \min\{n \in \mathbb{Z} \mid v(a_n) + nt = H(f,t)\},$$
$$n_{f,t}^- = \max\{n \in \mathbb{Z} \mid v(a_n) + nt = H(f,t)\}, \quad (t = v(z)).$$

If there are infinitely many $a_n \neq 0$ with $n < 0$, then

$$\lim_{t\to+\infty} n_{f,t}^+ = \lim_{t\to+\infty} n_{f,t}^- = -\infty.$$

Similarly, if there are infinitely many $a_n \neq 0$ with $n > 0$, then

$$\lim_{t\to-\infty} n_{f,t}^+ = \lim_{t\to-\infty} n_{f,t}^- = +\infty.$$

For each $t \in \mathbb{R}$ we set

$$h^+_{f,t} = n^+_{f,t}.t, \quad h^-_{f,t} = n^-_{f,t}.t \quad \text{and} \quad h_{f,t} = h^-_{f,t} - h^+_{f,t}.$$

Definition 2. *We call $h^+_{f,t}$, $h^-_{f,t}$, and $h_{f,t}$ the right local height, left local height and local height of the function $f(z)$ at $t = v(z)$, respectively.*

Lemma 1.

a) *If $h_{f,t} = 0$, then $f(z) \neq 0$ for z satisfying $v(z) = t$, and we have $\|f(z)\| = p^{-H(f,t)}$.*

b) *If $h_{f,t} \neq 0$, then $f(z)$ has zeros at $v(z) = t$, and $h_{f,t}$ is equal to $t\times$* Card *{the number of zeros of $f(z)$ at $v(z) = t$}.*

c) *In every compact interval of $\mathbb{R}$ there are only finitely many t such that $h_{f,t} \neq 0$, and such a point t is called a critical point of $f(z)$.*

The proof of the lemma follows from definition of the notion of height and the properties of the Newton polygon of $f(z)$.

Let $f(z)$ be a holomorphic function on $\mathbb{C}_p^*$,

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n.$$

We set

$$f_+(z) = \sum_{n=0}^{+\infty} a_n z^n, \quad f_-(z) = \sum_{n=1}^{+\infty} a_{-n} z^n.$$

Then $f_+(z)$, $f_-(z)$ are holomorphic functions in $\mathbb{C}_p$, and

$$f(z) = f_+(z) + f_-(\frac{1}{z}).$$

From the geometric interpretation of heights we can see that

$$H(f,t) = \min\{H(f_+,t), H(f_-,-t)\} \quad \text{for every} \quad t \in \mathbb{R},$$

and there exists $t_0 \in \mathbb{R}$ such that $H(f_+,t_0) = H(f_-,-t_0)$. We have

Lemma 2.

$$1) \quad H(f,t) = \begin{cases} H(f_+,t), & \text{if} \quad t \leq t_0, \\ H(f_-,-t), & \text{if} \quad t > t_0. \end{cases}$$

$$2) \quad h^+_{f,t} = \begin{cases} h^+_{f_+,t}, & \text{if} \quad t \leq t_0, \\ h^-_{f_-,-t}, & \text{if} \quad t > t_0. \end{cases}$$

$$3) \quad h^-_{f,t} = \begin{cases} h^-_{f_+,t}, & \text{if} \quad t \leq t_0, \\ h^+_{f_-,-t}, & \text{if} \quad t > t_0. \end{cases}$$

$$4) \quad h_{f,t} = \begin{cases} h_{f+,t}, & \text{if} \quad t \leq t_0, \\ -h_{f_-,t}, & \text{if} \quad t > t_0. \end{cases}$$

Proof

1) Since for every $n \in \mathbb{Z}$, $n > 0$ the graphs $\Gamma_n = v(a_{-n}) + nt$ and $\Gamma_n' = v(a_{-n}) - nt$ are symmetric for each other through the axis of ordinates, if t is a critical point of $f(z)$, $t > t_0$, then $-t$ is a critical point of $f_-(z)$. Therefore if $t > t_0$, then $H(f,t) = H(f_-, -t)$.

In case $t \leq t_0$, by the properties of the Newton polygon we have

$$H(f_+, t) < H(f_+, t_0) = H(f_-, -t_0) < H(f_-, -t).$$

Hence, $H(f,t) = H(f_+, t)$.

2), 3), 4) For $t \leq t_0$, since $H(f,t) = H(f_+, t)$, we have

$$n^+_{f,t} = n^+_{f_+,t}, \quad n^-_{f,t} = n^-_{f_+,t}.$$

Consequently,

$$h^+_{f,t} = h^+_{f_+,t}, \quad h^-_{f,t} = h^-_{f_+,t}, \quad h_{f,t} = h_{f_+,t}.$$

For $t > t_0$, since $H(f,t) = H(f_-, -t)$, we have

$$n^-_{f,t} = -n^+_{f_-,-t}, \quad n^+_{f,t} = -n^-_{f_-,-t}.$$

Consequently,

$$h^-_{f,t} = h^+_{f_-,-t}, \quad h^+_{f,t} = h^-_{f_-,-t}, \quad h_{f,t} = -h_{f_-,-t}.$$

The lemma is proved. ■

Now we give an analog of the Poisson-Jensen formula for p-adic holomorphic functions on $\mathbb{C}_p^*$.

Theorem 1. *Let $f(z)$ be a p-adic holomorphic function on $\mathbb{C}_p^*$, and assume that $H(f_+, t_0) = H(f_-, -t_0)$ for a $t_0 \in \mathbb{R}$. Then*

1) *If $t \leq t_0$, then $H(f,t_0) - H(f,t) = h^-_{f,t_0} - h^+_{f,t} + \sum_{t<s<t_0} h_{f,s}$.*
2) *If $t' > t_0$, then $H(f,t_0) - H(f,t') = h^+_{f,t_0} - h^-_{f,t'} - \sum_{t_0<s<t'} h_{f,s}$.*
3) *If $t < t_0 < t'$, then $H(f,t) - H(f,t') = (h^+_{f,t_0} - h^-_{f,t_0}) + (h^+_{f,t} - h^-_{f,t'}) - \sum_{t_0<s<t'} h_{f,s} - \sum_{t<s<t_0} h_{f,s}$.*

Proof. Let $t' \geq t_{-m} > \cdots > t_{-1} > t_0 > t_1 > ... > t_n \geq t$ be all the critical points of $f(z)$ in $[t,t'] \subset \mathbb{R}$.

1) For $k = 0, 1, 2, ...$ we have

$$n^-_{f,t_k} = n^+_{f,t_{k+1}}, \quad \text{and} \quad H(f,t_k) - H(f,t_{k+1}) = n^-_{f,t_k}(t_k - t_{k+1}).$$

Therefore, for $t \leq t_0$,

$$\begin{aligned}
H(f,t_0) - H(f,t) &= n^-_{f,t_0}(t_0 - t_1) + n^-_{f,t_1}(t_1 - t_2) + \cdots + n^-_{f,t_n}(t_n - t) \\
&= n^-_{f,t_0}.t_0 + \sum_{i=1}^{n}(n^-_{f,t_i} - n^-_{f,t_{i+1}}).t_i - n^-_{f,t_n}.t \\
&= h^-_{f,t_0} - h^+_{f,t} + \sum_{t \leq s < t_0} h_{f,s}.
\end{aligned}$$

2) Similarly, for $k = 1, 2, 3, ...$ we have

$$n^+_{f,t_{-k}} = n^-_{f,t_{-(k+1)}} \quad \text{and} \quad H(f, t_{-k}) - H(f, t_{-(k+1)}) = n^+_{f,t_{-k}}(t_{-k} - t_{-(k+1)}).$$

Thus,

$$\begin{aligned} H(f, t_0) - H(f, t') &= n^+_{f,t_0}(t_0 - t_{-1}) + \cdots + n^+_{f,t_{-m}}(t_{-m} - t') \\ &= n^+_{f,t_0}.t_0 - \sum (n^-_{f,t_{-k}} - n^+_{f,t_{-k}}).t_{-k} - n^-_{f,t'}.t' \\ &= h^+_{f,t_0}.t_0 - h^-_{f,t'} - \sum_{t_0<s<t'} h_{f,t}. \end{aligned}$$

3) is a consequence of 1) and 2). ■

Now we let $U = \{..., u_{-l}, \ldots, u_{-1}, u_0, u_1, u_2, \ldots, u_n, ...\}$ be a sequence of distinct points on $\mathbb{C}^*_p$ satisfying $v(u_i) \geq v(u_{i+1})$ for all $i \in \mathbb{Z}$. We shall consider only sequences U such that the number of points u_i in every $C(\mu, \nu)$ is finite for $-\infty < \mu < \nu < +\infty$. We set

$$n^+_{U,t} = \begin{cases} \#\{u_i \in U \mid \ t < v(u_i) \leq t_0\}, & \text{if } \ t \leq t_0, \\ \#\{u_i \in U \mid \ t_0 < v(u_i) < t\}, & \text{if } \ t > t_0. \end{cases}$$

$$n^-_{U,t} = \begin{cases} \#\{u_i \in U \mid t \leq v(u_i) \leq t_0\}, & \text{if } \ t \leq t_0, \\ \#\{u_i \in U \mid \ t_0 < v(u_i) \leq t\}, & \text{if } \ t > t_0, \end{cases}$$

where $t_0 = v(u_0)$.

We can define *the height* $H_{U,t}$ *of* U as follows.

Definition 3. *For each* $t \in \mathbb{R}$,

$$h^+_{U,t} = n^+_{U,t}.t, \quad h^-_{U,t} = n^-_{U,t}.t, \quad h_{U,t} = h^-_{U,t}.t - h^+_{U,t},$$

and

$$H(U, t) = \begin{cases} h^+_{U,t} - h^-_{U,t_0} - \sum_{s>t} h_{U,s}, & \textit{if} \quad t \leq t_0, \\ h^-_{U,t_0} - h^+_{U,t} + \sum_{s<t} h_{U,s}, & \textit{if} \quad t > t_0. \end{cases}$$

We shall always assume that $\lim_{|t|\to\infty} H(U, t) = -\infty$.

We set

$$U_+ = \{u_0, u_1, u_2, \ldots\},$$

where $v(u_0) \geq v(u_1) \geq ...$, and set

$$U_- = \Big\{\frac{1}{u_{-1}}, \frac{1}{u_{-2}}, \ldots\Big\},$$

where $v\Big(\frac{1}{u_{-1}}\Big) \geq v\Big(\frac{1}{u_{-2}}\Big) \geq ...$

For $t \leq t_0$, we have $n^+_{U,t} = n^+_{U^+,t}$, $n^-_{U,t} = n^-_{U^+,t}$.

Consequently,

$$h^+_{U,t} = h^+_{U_+,t}, \quad h^-_{U,t} = h^-_{U_+,t}, \quad h_{U,t} = h_{U_+,t}, \quad \text{and} \quad H(U,t) = H(U_+,t).$$

For $t > t_0$ we have $n^+_{U,t} = n^+_{U_-,-t}$, $\quad n^-_{U,t} = n^-_{U_-,t}$, and then

$$h^+_{U,t} = -h^+_{U_-,-t}, \quad h^-_{U,t} = -h^-_{U_-,-t}, \quad h_{U,t} = -h_{U_-,-t}, \quad \text{and} \quad H(U,t) = H(U_-,-t).$$

3. Interpolation of Holomorphic Functions on $\mathbb{C}_p^*$

Now we shall use the notion of height of a holomorphic function and a sequence to give an interpolation theorem. We first need the following definitions.

Definition 4. *Let $f(z)$ be a holomorphic function on $\mathbb{C}_p$. A sequence $U = \{u_i\}_{i=0}^{\infty}$ is called an interpolating sequence of $f(z)$ if the sequence of interpolation polynomals for f on U converges to $f(z)$.*

Definition 5. *Let $f(z)$ be a holomorphic function on $\mathbb{C}_p^*$. A sequence $U = \{u_i\}_{i=-\infty}^{+\infty}$ is called an interpolation sequence of $f(z)$ if U_+ is an interpolating sequence of $f_+(z)$ and U_- is an interpolating sequence of $f_-(z)$.*

Theorem 2. *Let $f(z)$ be a holomorphic function on $\mathbb{C}_p^*$. A sequence U is an interpolating sequence of $f(z)$ if and only if*

$$\lim_{|t|\to\infty} \{H(f,t) - H(U,t)\} = +\infty$$

Proof. We first prove the necessary condition.

Let $\{P_k(z)\}$ be the sequence of Lagrange interpolation polynomials for $f_+(z)$ on U_+. We have

$$\deg P_k(z) \le k, \quad P_k(u_i) = f_+(u_i), \quad i = 0, 1, 2, ...k.$$

We set $\quad S_k(z) = P_{k+1}(z) - P_k(z)$.

By the assumption we have

$$f_+(z) = \sum_{k=0}^{\infty} S_k(z).$$

This implies

$$H(f_+,t) \ge \min_{k\ge 0}\{H(S_k,t)\}.$$

Thus,

$$H(f_+,t) - H(U_+,t) \ge \min_{k\ge 0}\{H(S_k,t) - H(U_+,t)\}.$$

Since S_k is a polynomial of degree k and $\lim\limits_{t\to-\infty} H(U_+,t) = -\infty$,

$$\lim_{t\to-\infty}\{H(S_k,t) - H(U_+,t)\} = +\infty.$$

It follows that

$$\lim_{t\to-\infty}\{H(f_+,t)-H(U_+,t)\}=+\infty. \tag{1}$$

We also have a similar result for the function f_- and the sequence U_-, i.e.,

$$\lim_{t\to+\infty}\{H(f_-,-t)-H(U_-,-t)\}=+\infty. \tag{2}$$

On the other hand, for $t\ll 0$ we have

$$H(f,t)=H(f_+,t) \quad \text{and} \quad H(U,t)=H(U_+,t),$$

for $t\gg 0$,

$$H(f,t)=H(f_-,-t) \quad \text{and} \quad H(U,t)=H(U_-,-t).$$

By combining these equalities with (1) and (2) we obtain

$$\lim_{|t|\to\infty}\{H(f,t)-H(U,t)\}=+\infty.$$

The necessity is proved. ∎

Now we prove the sufficiency. By using arguments similar to ones used in the proof of Lemma 3.10 in [6] we obtain the following

Lemma 3. *For k such that $v(u_k)=t_k<t_{k_0}<0$ we have*

$$[H(S_k,t_{k_0})-H(U_+,t_{k_0})]-[H(S_k,t_k)-H(U_+,t_k)]\geq t_{k_0}.$$

Let $\{t_k\}$ be a sequence of real numbers, $t_k\to-\infty$ as $k\to+\infty$. There exists k_0 such that for all $k>k_0,\ N>k_0,\ \ t_k=v(u_k)<t_N<0$. We have

$$H(S_k,t_N)\geq H(U_+,t_N)+[H(S_k,t_k)-H(U_+,t_k)]+t_N.$$

Further, by Lemma 3.11 in [6] we get

$$\begin{aligned}&H(S_k,t_N)\geq H(U_+,t_N)\\&+t_N+\min\{[H(f_+,t_k)-H(U_+,t_k)],[H(f_+,t_{k+1})-H(U_+,t_{k+1})]\}.\end{aligned}$$

From this and the assumption we have

$$\lim_{k\to+\infty}H(S_k,t_N)=+\infty.$$

In other words, $\lim\limits_{k\to+\infty}S_k(z)=0$. Hence, the sequence $\{P_k(z)\}$ converges.

Let $P(z)=\lim\limits_{k\to+\infty}P_k(z)$. We show that $P(z)=f_+(z)$.

Set $g(z)=P(z)-f_+(z)$. We have $g(u_i)=0$ for $i=0,1,...$ thus,

$$n^-_{g,t}\geq n^-_{U_+,t} \quad \text{for every} \quad t\in\mathbb{R}. \tag{3}$$

On the other hand, by Lazard's Lemma [9] we have

$$f_+(z)=\phi(z)\prod_{i=0}^{k}(z-u_i)+Q_k(z),$$

where $\deg Q_k(z) \leq k$, $Q_k(u_i) = P_k(u_i)$ for $i = 0, 1, 2...k$, $H(Q_k, t) \geq H(f_+, t)$.

Hence we have $Q_k(z) \equiv P_k(z)$. Thus, $H(P_k, t) \geq H(f_+, t)$, and consequently, $H(P, t) \geq H(f_+, t)$.

We have

$$\lim_{t \to -\infty} [H(g, t) - H(U_+, t)] = +\infty. \tag{4}$$

This implies $g(z) \equiv 0$, because if $g(z) \not\equiv 0$ we see that (3) constradicts (4).

It follows that U_+ is an interpolation sequence of $f_+(z)$. Similarly, U_- is an interpolation sequence of $f_-(z)$.
The proof of Theorem 2 is complete. ∎

Now let $f(z)$ be a holomorphic function on $\mathbb{C}_p$ represented by a Taylor series $f(z) = \sum_{n=0}^{+\infty} a_n z^n$. From Theorem 2 we obtain the following

Corollary. *A sequence $U = \{u_0, u_1, u_2, ...\}$ for which $v(u_i) \geq v(u_{i+1})$ and the number of points u_i satisfying $v(u_i) \geq t$ is finite for $t \in \mathbb{R}$, is an interpolation sequence of $f(z)$ if and only if*

$$\lim_{t \to -\infty} \{H(f, t) - H(U, t)\} = +\infty.$$

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