Vietnam Journal of Mathematics **31**:2 (2003) 185–192

Vietnam Journal of MATHEMATICS

Autoregressive Time Series Are L_p-Mixingales

Dao Quang Tuyen

Hanoi Institute of Mathematics, P.O. Box 631 Bo Ho, Hanoi, Vietnam

Received August 18, 2001

Abstract. Autoregressive time series, generated by data which are α-mixing (strong mixing), φ-mixing or are L_p-mixingales, are proved to be L_p-mixingales. Under a condition on mixingale rates they satisfy the strong law.

1. Introduction

Let (Y_n) and (X_n), $n = 0, \pm 1, \pm 2, \dots$, be sequences of real random variables such that

$$X_n = \sum_{i=1}^{\infty} d_i X_{n-i} + Y_n \quad \text{a.s.,} \tag{1}$$

for all n, where d_i are constants such that $d = \sum_{i=1}^{\infty} |d_i| < 1$. Then we say $X = (X_n)$ is an *autoregressive time series* generated by random data $Y = (Y_n)$.

Even when Y_n are independent random variables defined by $P(Y_n = 1) = P(Y_n = 0) = 1/2$, the sequence (X_n) is not necessarily α-mixing (strong mixing)(cf. examples in Davidson [1, p.216]). It seems most autoregressive time series are not α-mixing i.e. X_n depends strongly on each other. So the known limit theorems, especially the strong law, for mixing random variables could not be applied directly to them. There are only few strong laws found for (X_n) (as far as author knows). McLeish [4, Example 2] and Peligrad [5, Application 2] showed that if (Y_n) is a martingale difference sequence then (X_n) satisfies the strong law under conditions on rates of tending to zero of $|d_n|$ and that $\sup_n \|X_n\|_2 < \infty$. Also, if Y_n are uniformly bounded φ-mixing (uniform mixing) random variables then the strong law holds also for (X_n) (cf. Tuyen [6]).

A sequence of real random variables Y_n is called α-mixing or φ-mixing if

$$\alpha_n = \sup_k \sup |P(BA) - P(B)P(A)| \to 0 \quad \text{or}$$

$$\varphi_n = \sup_k \sup |P(B|A) - P(B)| \to 0,$$

respectively, where both second *sup*'s are taken over all the sets $A \in \sigma(Y_1, ..., Y_k)$ and $B \in \sigma(Y_{k+n})$, such that $P(A)$ and $P(B) > 0$ (cf. McLeish [4]).

Let (Y_n), $n = 0, \pm 1, \pm 2, \ldots$, be any sequence of integrable random variables and $(\mathcal{F}_n)$ be any sequence of increasing sub σ-algebras of a σ-algebra $\mathcal{F}$. For simplicity denote $E_n Y_m = E(Y_m|\mathcal{F}_n)$. The sequence of pairs $(Y_n, \mathcal{F}_n)$ is called L_p*-mixingale* for any $p \geq 1$, if $\|Y_n\|_p < \infty$ for all n and there exist finite nonnegative numbers C_n, and ζ_m, $m = 0, 1...$, called *mixingale coefficients*, such that $\zeta_m \to 0$ and

$$\|E_{n-m} Y_n\|_p \leq C_n \zeta_m,$$
$$\|Y_n - E_{n+m} Y_n\|_p \leq C_n \zeta_{m+1}$$

hold for $m \geq 0$ and all n (cf. Davidson [1, p. 247]). L_2-mixingales are usually called *mixingales* which were first dealt with by McLeish [4].

Remark. The random variables of any L_p-mixingale are always zero mean-valued since, by notations above, for any n

$$|EY_n| = |E(E_{n-m} Y_n)| \leq \|E_{n-m} Y_n\|_p \leq C_n \zeta_m \to 0 \quad \text{as } \ m \to \infty.$$

All α-mixing or φ-mixing sequences of centered random variables Y_n's are L_p-mixingales with $\mathcal{F}_n = \sigma(Y_1, Y_2, \ldots, Y_n)$ if they are bounded by L_r-norm, i.e., $\sup_n \|Y_n\|_r < \infty$, for any $p < r \leq \infty$ [4, p. 834]. In this case $\zeta_n = \alpha_n^{1/p-1/r}$ or $\varphi_n^{1-1/r}$, respectively, and $C_n = 6 \sup_n \|Y_n\|_r$.

This paper shows that any sequence (X_n) of (1) is a L_p-mixingale after centered, if (Y_n) is so. As a consequence, any limit theorems for L_p-mixingales could apply to them. Moreover if mixingale coefficients ζ_n of data (Y_n) obey some rate conditions while tending to zero then so does the time series's. Then under rate conditions they are showed to satisfying the strong law.

2. Results

It seems most (X_n) of (1) are L_p-mixingales although are not α-mixing, including examples in [1], as shown by the following

Theorem 1. *Let* (X_n)*,* $n = 0, \pm 1, \pm 2, ...$*, be an autoregressive time series defined by* (1) *such that* $d < 1$*. Suppose for any* $p \geq 1$ $\limsup_{n \to -\infty} \|X_n\|_p < \infty$*,* $\sup_n \|Y_n\|_p < \infty$ *and* $(Y_n - EY_n)$ *is a* L_p*-mixingale such that* $C_n = C_n(Y)$ *is bounded. Then* $(X_n - EX_n)$ *is also a* L_p*-mixingale with the same sequence of* σ*-algebras* $\mathcal{F}_n = \mathcal{F}_n(Y)$ *and with* $C_n(X) = 1$*.*

One important class of (X_n) is the case when (X_n) has a starting point, that is $X_n = Y_n = 0$, for all $n < 0$, which leads to that (1) is a recursive formula. In that case if Y_n is $\mathcal{F}_n(Y)$-measurable, especially $\mathcal{F}_n(Y) = \sigma(Y_1, Y_2, ..., Y_n)$, we have $\sigma(X_1, X_2, ..., X_n) \subset \mathcal{F}_n(Y)$. Then the sequence of pair $(X_n - EX_n, \sigma(X_1, X_2, ..., X_n))$ is also a L_p-mixingale with the same mixingale coefficients, since by Jensen's inequality for conditional expectations, for any $m < n$

$$\|E[X_n - EX_n|\sigma(X_1, X_2, \dots, X_m)]\|_p \le \|E[X_n - EX_n|\mathcal{F}_m(Y)]\|_p.$$

In the case when some rate of tending to zero of mixingale coefficients of $(Y_n - EY_n)$ can be verified, the time series $(X_n - EX_n)$ has the same "mixingale rate".

Theorem 2. *Let (X_n) be a sequence satisfying all the conditions of Theorem 1. Suppose further that*

$$|d_n| = O(1/n^{q+1}L(n)) \quad \textit{and} \quad \zeta_n(Y) = O(1/n^q L(n)),$$

for any positive slowly varying sequence $L(n)$ and $0 < q < 1$, where $\zeta_n(Y)$ are mixingale coefficients of $(Y_n - EY_n)$. Then there exist also $\zeta_n(X)$, the mixingale coefficients of $(X_n - EX_n)$, such that

$$\zeta_n(X) = O(1/n^q L(n)).$$

Under a stronger rate condition (X_n) satisfies also the strong law. Let us first introduce a needed definition of *size of rates*.

Definition. [4] *A sequence of nonnegative numbers (a_n), $n \ge 1$, is called of size $-q$ for any $q > 0$ if there exists a positive sequence (L_n) such that*
(a) $\sum_{n=1}^{\infty} 1/nL_n < \infty$,
(b) $L_n - L_{n-1} = O(L_n/n)$,
(c) *there is a k such that L_n is nondecreasing on $\{n \ge k\}$ and*
(d) $a_n = O(1/n^q[L_n]^{2q})$.

Note that $a_n \to 0$ by (d) and (c), so the above definition deals really with rates of tending to zero of sequence.

For definiteness, we also call a sequence a_n, $n \ge 1$ *slowly varying* if its extension $L(t) := a_n$ for $n - 1 < t \le n$ on the interval $(0, \infty)$ is a slowly varying function (cf. Feller [3, p. 269]).

Theorem 3. *Let (X_n) be a sequence satisfying all the conditions of Theorem* 1. *Suppose $(|d_n|)$ is of size $-3/2$ with a slowly varying sequence $L_n(d)$, and $\{\zeta_n(Y)\}$ is of size $-1/2$ with a slowly varying sequence $L_n(Y)$. Then there exists a slowly varying sequence L_n such that $(X_n - EX_n)$ is a L_p-mixingale with $\{\zeta_n(X)\}$ of size $-1/2$ with L_n. Consequently for $p = 2$ $\sum_{i=1}^{n}(X_i - EX_i)/n \to 0$ almost surely, as $n \to \infty$.*

3. Proofs

Lemma 1. *Given nonnegative numbers a_n, ζ_n, x_n, $n = 1, 2, \dots$ and $x_0 > 0$ such that $\sum_{n=1}^{\infty} a_n = a < 1$ and $\zeta_n \to 0$. Suppose*

$$x_n \le \sum_{i=n+1}^{\infty} a_i x_0 + \sum_{i=1}^{n} a_i x_{n-i} + \zeta_n \tag{2}$$

for all $n \ge 1$. Then $x_n \to 0$.

If in addition $a_n = O(1/n^{q+1}L(n))$ and $\zeta_n = O(1/n^q L(n))$ for any $0 < q < 1$, where $L(n)$ is any given positive slowly varying sequence, then $x_n = O(1/n^q L(n))$.

Proof. Put $\zeta'_n = \sup_{i \ge n} \zeta_i$ for $n \ge 1$. Then $\zeta'_n \downarrow 0$ and $\zeta'_n \ge \zeta_n$. Put $B = \max(x_0, \zeta'_1/(1-a))$. Define recursively $v_1 = aB + \zeta'_1$ and

$$v_n = \sum_{i=n}^{\infty} a_i B \; + \; \sum_{i=1}^{n-1} a_i v_{n-i} + \zeta'_n,$$

for $n \ge 2$. Then $v_1 \le B$ and by induction $x_n \le v_n$, for all n. Hence it is enough to show $v_n \to 0$.

Since

$$v_n - v_{n+1} = a_n B - a_n v_1 + \sum_{i=1}^{n-1} a_i (v_{n-i} - v_{n+1-i}) + (\zeta'_n - \zeta'_{n+1}),$$

for $n \ge 2$ and since $v_2 \le v_1 \le B$, by induction $v_n - v_{n+1} \ge 0$, for all n. So $B \ge v_n \downarrow v \ge 0$. For any $\varepsilon > 0$ there exists $n(\varepsilon)$ such that $v \le v_n \le v + \varepsilon$, for all $n \ge n(\varepsilon)$. Then by the definition of v_n, for $n \ge n(\varepsilon)$

$$v \le v_n \le \sum_{i=n-n(\varepsilon)}^{\infty} a_i B + \sum_{i=1}^{n-n(\varepsilon)-1} a_i (v+\varepsilon) + \zeta'_n.$$

Letting $n \to \infty$ we see that the last term tends to $a(v+\varepsilon)$. So $v \le a(v+\varepsilon)$. As $a < 1$, v must be 0.

For the second part of Lemma 1 let C be a constant such that $a_n \le C/n^{q+1} L(n)$ and $\zeta_n \le C/n^q L(n)$ for all n. Since $C/n^{q+1}L(n)$ is summable [3, Lemma, p. 272], there exists N such that $\sum_{n=N}^{\infty} C/n^{q+1}L(n) < 1 - a$. Define

$$f_n = \begin{cases} a_n, & \text{for } n < N \\ C/n^{q+1}L(n), & \text{otherwise.} \end{cases}$$

Then $a_n \le f_n$ for all n and still $\sum_1^{\infty} f_n < 1$. By [3, Theorem 1, p. 273] of Feller and that $q > 0$

$$\zeta_n \le C/n^q L(n) \le C_1 \sum_{i=n}^{\infty} f_i, \tag{3}$$

for some constant C_1 and large enough n, hence for all n with large enough C_1.

For such f_n define recursively $u_o := x_0 + C_1$ and for $n \ge 1$

$$u_n := \sum_{i=n+1}^{\infty} f_i u_o + \sum_{i=1}^{n} f_i u_{n-i} = \sum_{i=n}^{\infty} f_i u_o + \sum_{i=1}^{n-1} f_i u_{n-i}. \tag{4}$$

By (3) and that $a_n \le f_n$

$$\sum_n^\infty a_i x_0 + \zeta_n \le \sum_n^\infty f_i u_o,$$

for all $n \ge 1$. Hence rewritting (2) in the form

$$x_n \le \sum_{i=n}^\infty a_i x_0 + \sum_{i=1}^{n-1} a_i x_{n-i} + \zeta_n,$$

we get by induction $x_n \le u_n$, for all n. It remains to show that the conclusion holds for u_n, for which we follow the proof of McLeish [4, p. 833].

First note that by (4) u_n is nonnegative and monotone non-increasing since $u_n - u_{n+1} = \sum_{i=1}^n f_i(u_{n-i} - u_{n+1-i})$ for $n \ge 1$, and $u_o \ge u_1$, and then use induction.

Since (4) is the renewal equation (cf. Feller [2, p. 330]) hence if we define generating functions $F(s) := \sum_{i=1}^\infty f_i s^i$, $V(s) := \sum_{i=o}^\infty u_i s^i$ and $B(s) := \sum_{i=o}^\infty b_i s^i$ where $b_o := u_o$, $b_n := \sum_{i=n+1}^\infty f_i u_o$ for $n \ge 1$, for any $0 < s < 1$, then by some elementary computations we have, for each $0 < s < 1$,

$$B(s) = u_o(1 - F(1)) + u_o \frac{F(1) - F(s)}{1 - s}$$

and

$$V(s) \;=\; \frac{B(s)}{1 - F(s)}.$$

Since $F(s) \uparrow F(1) < 1$, $V(s) \sim (\text{constant})B(s)$, as $s \uparrow 1$. By Theorem 1 of Feller [3, p. 273] $b_n \sim \text{constant}/n^q L(n)$. Again by that theorem, as $q < 1$, $\sum_{i=o}^n b_i \sim (\text{constant})n^{1-q}/L(n)$. Consequently by Theorem 5 of Feller [3, p. 423] applying to $B(s)$

$$V(s) \sim (\text{constant})B(s) \sim \frac{\text{constant}}{(1-s)^{1-q} L(1/(1-s))},$$

as $s \uparrow 1$. Now because u_n is monotone and $q < 1$, applying the reverse part of the last cited theorem to $V(s)$ we can conclude that $u_n \sim \text{constant}/n^q L(n)$. ∎

For x_n satisfying a non-recurrent relation (5) below, we will have the same conclusions.

Lemma 2. *For nonnegative numbers a_n, ζ_n, x_n, for $n \ge 1$, and $B > 0$ suppose $\sum_{n=1}^\infty a_n = a < 1$, $\zeta_n \to 0$, $x_n < B$ and*

$$x_n \le \sum_{i=n+1}^\infty a_i B + \sum_{i=1}^n a_i x_{n+i} + \zeta_n, \tag{5}$$

for all n. Then $x_n \to 0$.

If in addition $a_n = O(1/n^{q+1}L(n))$ and $\zeta_n = O(1/n^q L(n))$ for any $q > 0$, where $L(n)$ is any given positive slowly varying sequence, then $x_n = O(1/n^q L(n))$.

Proof. To prove $x_n \to 0$, suppose, in contrary, $\limsup_n x_n = A > 0$, where $A \le B$. Then for any $\varepsilon > 0$, there exists $n(\varepsilon)$ for all $n > n(\varepsilon)$ $x_n < A+\varepsilon$, $\zeta_n < \varepsilon$ and $\sum_{i=n+1}^{\infty} a_i B < \varepsilon$. So by (5), as $a < 1$, for suitable chosen small $\varepsilon > 0$ and all $n > n(\varepsilon)$

$$x_n < \sum_{i=n+1}^{\infty} a_i B + \sum_{i=1}^{n} a_i(A+\varepsilon) + \varepsilon < a(A+\varepsilon) + 2\varepsilon < A - \varepsilon,$$

which leads to contradiction with $\limsup_n x_n = A$.

For the remaining part of Lemma 2, as in the proof of Lemma 1, we can find f_n such that $f_n = O(1/n^{q+1}L(n))$, $a_n \le f_n$ for all $n \ge 1$, $\sum_1^{\infty} f_n = f < 1$ and

$$\sum_{n}^{\infty} a_i B + \zeta_n \le \sum_{n}^{\infty} f_i B' \sim \text{constant}/n^q L(n),$$

for some $B' \ge B$ and all $n \ge 1$.

Put $v_n = \sup_{i \ge n} x_i$. Then $v_n \downarrow$ and $x_n \le v_n \le B$ for all n. By (5) for any $k \ge n$

$$x_k \le \sum_{i=k}^{\infty} a_i B + \sum_{i=1}^{k-1} a_i x_{k+i} + \zeta_k \le \sum_{i=k}^{\infty} f_i B' + \sum_{i=1}^{k-1} f_i v_{n+i} \le \sum_{i=n}^{\infty} f_i B' + \sum_{i=1}^{n-1} f_i v_{n+i}.$$

Taking the supremum over the set $\{k : k \ge n\}$ on both sides we have

$$v_n \le \sum_{i=n}^{\infty} f_i B' + \sum_{i=1}^{n-1} f_i v_{n+i} \le \sum_{i=n}^{\infty} f_i B' + f v_n,$$

since $v_n \downarrow$. Hence $x_n \le v_n \le \frac{B'}{1-f} \sum_{i=n}^{\infty} f_i$. ■

Proof of Theorem 1. Without loss of generality we can suppose $\|Y_n\|_p \le B$, $C_n(Y) \le B$ for all n and $\|X_n\|_p \le \frac{B}{1-d}$ for $n \le K$.

Since $X_{n,k} := \sum_{i=1}^{k} d_i X_{n-i} + Y_n \to X_n$ a.s. as $k \to \infty$, applying Fatou's lemma to $|X_{n,k}|^p$ we have

$$\|X_n\|_p \le \liminf_k \|X_{n,k}\|_p \le \sum_{i=1}^{\infty} |d_i| \|X_{n-i}\|_p + \|Y_n\|_p.$$

Then by induction, begin at K, for all n

$$\|X_n\|_p \le \frac{B}{1-d}. \tag{6}$$

Furthermore $|X_{n,k}| \le \sum_{i=1}^{\infty} |d_i||X_{n-i}| + |Y_n|$, where the last term is integrable by Monotone convergence theorem and (6). So by Lebesgue dominated convergence theorem $EX_{n,k} \to EX_n$. Hence we have

$$X_n - EX_n = \sum_{i=1}^{\infty} d_i(X_{n-i} - EX_{n-i}) + (Y_n - EY_n) \quad \text{a.s.},$$

that is (1) holds also for centered random variables. So we suppose from now on, without loss of generality, $EY_n = EX_n = 0$, for all n.

Also by Lebesgue dominated convergence theorem for conditional expectations we have $E_{n-N}X_{n,k} \to E_{n-N}X_n$ a.s., for all $N \geq 1$. So applying Fatou lemma for conditional expectations to $|E_{n-N}X_{n,k}|^p$ and using (6) we have

$$\|E_{n-N}X_n\|_p \leq \liminf_k \|E_{n-N}X_{n,k}\|_p$$

$$\leq \sum_{i=1}^{\infty} |d_i| \|E_{n-N}X_{n-i}\|_p + \|E_{n-N}Y_n\|_p$$

$$\leq \sum_{i=N}^{\infty} |d_i| \frac{B}{1-d} + \sum_{i=1}^{N-1} |d_i| \|E_{n-N}X_{n-i}\|_p + \|E_{n-N}Y_n\|_p.$$

Taking the supremum over n on both sides and putting $\zeta'_N(X) = \sup_n \|E_{n-N}X_n\|_p$ we obtain that

$$\zeta'_N(X) \leq \sum_{i=N}^{\infty} |d_i| \frac{B}{1-d} + \sum_{i=1}^{N-1} |d_i| \zeta'_{N-i}(X) \; + \; B\zeta_N(Y)$$

for all $N \geq 1$, where $\zeta_n(Y)$ are mixingale coefficients of (Y_n). Putting $\zeta'_o(X) := \frac{B}{1-d}$, as $d < 1$ and $\zeta_N(Y) \to 0$, by Lemma 1 $\zeta'_N(X) \to 0$.

Similarly we have for all $N \geq 1$ $E_{n+N}X_{n,k} \to E_{n+N}X_n$ a.s. and

$$\|E_{n+N}X_n - X_n\|_p \leq \liminf_k \|E_{n+N}X_{n,k} - X_{n,k}\|_p$$

$$\leq \sum_{i=n+1}^{\infty} |d_i| \frac{2B}{1-d} + \sum_{i=1}^{n} |d_i| \|E_{n+N}X_{n-i} - X_{n-i}\|_p + \|E_{n+N}Y_n - Y_n\|_p.$$

Hence putting $\zeta''_N(X) = \sup_n \|E_{n+N}X_n - X_n\|_p$ we obtain that

$$\zeta''_N(X) \leq \sum_{i=n+1}^{\infty} |d_i| \frac{2B}{1-d} + \sum_{i=1}^{n} |d_i| \zeta''_{N+i}(X) + B\zeta_{N+1}(Y).$$

As by (6) $\zeta''_n(X) \leq \frac{2B}{1-d}$ for all n, by Lemma 2 $\zeta''_n(X) \to 0$.

So (X_n) is a L_p-mixingale with $\zeta_n(X) = \max(\zeta'_n(X), \zeta''_{n-1}(X))$ and $C_n(X) = 1$, according to the definition of L_p-mixingales. ∎

Proof of Theorem 2. The second parts of Lemma 1 and Lemma 2 applying to the inequalities for $\zeta'_N(X)$ and $\zeta''_N(X)$ in the proof of Theorem 1 lead to the conclusion, with the same $\zeta_n(X)$ defined above. ∎

Proof of Theorem 3. First note that, since $L_n(d)$ is positive eventually nondecreasing, there exists $a > 0$ such that $L_n(d) > a$ for large enough n, so by (d) of the definition of size rate

$$|d_n| \leq O(1/n^{\frac{3}{2}} a^2 L_n(d)) = O(1/n^{\frac{3}{2}} L_n(d)). \tag{7}$$

Define L_n such that

$$\frac{1}{L_n} = \frac{1}{L_n(d)} + \frac{1}{L_n(Y)}, \tag{8}$$

for all n. Then L_n is slowly varying, because sums and reciprocals of slowly varying functions are so.

As $L_n \le L_n(d)$ and $L_n \le L_n(Y)$, by (7) and (d) $|d_n| = O(1/n^{\frac{3}{2}} L_n)$ and $\zeta_n(Y) = O(1/n^{\frac{1}{2}} L_n)$, thus by Theorem 2 there exist $\zeta_n(X) = O(1/n^{\frac{1}{2}} L_n)$ such that $(X_n - EX_n)$ is a L_p-mixingale with them.

The sequence $\zeta_n(X)$ is of size $-(1/2)$, because beside (d) L_n satisfy (a) and (c) obviously. From (8), and for (b)

$$\begin{aligned} L_n - L_{n-1} &= L_n L_{n-1} \Big[\big(\frac{1}{L_{n-1}(d)} - \frac{1}{L_n(d)}\big) + \big(\frac{1}{L_{n-1}(Y)} - \frac{1}{L_n(Y)}\big) \Big] \\ &\le L_n L_{n-1} \Big[\frac{C}{nL_{n-1}(d)} + \frac{C}{nL_{n-1}(Y)} \Big] = \frac{C}{n} L_n, \end{aligned}$$

for some constant C, as $L_n(d)$ and $L_n(Y)$ satisfy (b).

So if in addition (X_n) and (Y_n) satisfy all the conditions of Theorem 1 with $p = 2$, the strong law holds for $(X_n - EX_n)$ according to Corollary 1.9 of McLeish [4]. ■

References

1. J. Davidson, *Stochastic Limit Theory*, Oxford Univ. Press, 1994.
2. W. Feller, *An Introduction to Probability Theory and Its Applications* **1**, 3rd edition, Wiley, New York, 1968.
3. W. Feller, *An Introduction to Probability Theory and Its Applications* **2**, 2nd edition, Wiley, New York, 1966.
4. D. L. McLeish, A maximal inequality and dependent strong laws, *Ann. Prob.* **3** (1975) 829–839.
5. M. Peligrad, Limit theorems and law of large numbers for martingale-like sequences, *Math. Nachr.* **99** (1980) 211–216.
6. D. Q. Tuyen, A strong law for mixing random variables, *Periodica Math. Hung.* **38** (1999) 131–136.