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The Inhomogeneous Generalized Cauchy–Riemann Systems in a Clifford Algebra

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Abstract. This paper deals with the inhomogeneous systems of the form $D_{x^{(h)}}g = f_h$ when f_h have compact supports. It shows the necessary and sufficient condition such that the mentioned systems have solutions with compact supports.

1. Preliminaries

Let $\mathbb{R}^m$ be a m-dimensional Euclidean space and $e_1, ..., e_m$ be an orthonormal basis of $\mathbb{R}^m$. Then a basis of $\mathcal{A}$ is given by

$$\{e_A : A \subset \mathcal{N};\ \mathcal{N} = \{1, 2, ..., m\}\}$$

where $e_i = e_{\{i\}},\ i = 1, ..., m,\ e_0 = e_\emptyset = 1$ and $e_A = e_{\alpha_1} \cdots e_{\alpha_h},\ A = \{\alpha_1, ..., \alpha_h\}$ with $1 \le \alpha_1 < \cdots < \alpha_h \le m$. The product in $\mathcal{A}$ is determined by the relations

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0 \quad \text{for} \quad i, j = 1, ..., m$$

where $e_0 = 1$ is the identity of $\mathcal{A}$. We define an involution in $\mathcal{A}$ as follows. Let a be an element of $\mathcal{A}$ which can be presented as

$$a = \sum_A a_A e_A, \quad a_A \in \mathbb{R}.$$

Put $\overline{a} = \sum_A a_A \overline{e}_A$ where $\overline{e}_A = \overline{e}_{\alpha_h} \overline{e}_{\alpha_{h-1}} \cdots \overline{e}_{\alpha_1}$,

$$\overline{e}_j = -e_j \quad \text{for} \quad j = 1, ..., m; \quad \overline{e}_0 = e_0.$$

For $n \le m$, the space $\mathbb{R}^{n+1}$ can be embedded in $\mathcal{A}$ as follows.

Hence $x = (x_0, x_1, ..., x_n)$ will be identified with

$$x_0 + \vec{x} = x_0 e_0 + \sum_{j=1}^{n} x_j e_j.$$

For other definitions, we refer the reader to [1]. Let Ω be an open set of the Euclidean space

$$\underbrace{\mathbb{R}^{m+1}(x^{(1)}) \times \mathbb{R}^{m+1}(x^{(2)}) \times \cdots \times \mathbb{R}^{m+1}(x^{(n)})}_{n} = \mathbb{R}^{(m+1)n}.$$

We consider the functions f, defined in Ω and taking values in the Clifford algebra $\mathcal{A}$. Then

$$f = \sum_{A} f_A(x^{(1)}, ..., x^{(n)}) e_A \quad \text{where} \quad x^{(j)} = (x_0^{(j)}, ..., x_m^{(j)}) \in \mathbb{R}^{m+1}(x^{(j)})$$

for $j = 1, ..., n$ and f_A are real-valued functions.

Further, we introduce the generalized Cauchy-Riemann operators

$$D_{x^{(j)}} = \sum_{i=0}^{m} e_i \frac{\partial}{\partial x_i^{(j)}}, \quad j = 1, ..., n$$

Definition 1.1. *A function $f : \Omega \to \mathcal{A}$ is called multi-regular in Ω if $f \in C^1(\Omega, \mathcal{A})$ and satisfies the systems*

$$D_{x^{(j)}} f = 0 \quad for \quad j = 1, ..., n. \tag{1.1}$$

Note that, in case $n = 1$, the multi-regular functions reduce to the regular functions (in sense of Delanghe(see [1])). Thus the multi-regular functions are the natural generalizations to higher dimensions of the regular functions.

In case $n = 2$ the multi-regular functions are quite different from the biregular functions studied by Brackx and Pincket (see [2]).

Remark 1.1. Let $f \in C^2(\Omega, \mathcal{A})$ be a multi-regular function, then

$$\Delta_{x^{(j)}} f = \overline{D}_{x^{(j)}} D_{x^{(j)}} f = D_{x^{(j)}} \overline{D}_{x^{(j)}} f = 0 \quad \text{for} \quad j = 1, ..., n.$$

Hence

$$\Delta f = \sum_{j=1}^{n} \Delta_{x^{(j)}} f = 0.$$

This means that f is harmonic in Ω. Then f is real analytic and the Uniqueness theorem is valid for the multi-regular functions f.

2. Some Properties of Functions Taking Values in the Clifford Algebra $\mathcal{A}$

Lemma 2.1. *Let Ω be an open set of $\mathbb{R}^{m+1}$ and $f, g \in C^1(\Omega, \mathcal{A})$. Then we have*

$$D_x(fg) = (D_x f)g + \sum_{i=0}^{m} e_i f \frac{\partial g}{\partial x_i}$$

where

$$D_x = \sum_{i=0}^{m} e_i \frac{\partial}{\partial x_i}, \quad x = (x_0, ..., x_m) \in \Omega. \tag{2.1}$$

Proof. Suppose that $f = \sum\limits_A f_A e_A, \quad g = \sum\limits_B g_B e_B$ then

$$fg = \sum_{A,B} f_A g_B e_A e_B. \tag{2.2}$$

We have

$$D_x(fg) = \sum_{i=0}^{m} e_i \frac{\partial (fg)}{\partial x_i}. \tag{2.3}$$

It is clear that

$$\begin{aligned}
\frac{\partial (fg)}{\partial x_i} &= \frac{\partial}{\partial x_i}\Big(\sum_{A,B} f_A g_B e_A e_B\Big) = \sum_{A,B} \frac{\partial (f_A g_B)}{\partial x_i} e_A e_B \\
&= \sum_{A,B} \Big(\frac{\partial f_A}{\partial x_i} g_B + \frac{\partial g_B}{\partial x_i} f_A\Big) e_A e_B = \sum_{A,B} \Big(\frac{\partial f_A}{\partial x_i} g_B e_A e_B + f_A \frac{\partial g_B}{\partial x_i} e_A e_B\Big) \\
&= \sum_{A,B} \Big(\frac{\partial f_A}{\partial x_i} e_A (g_B e_B) + f_A e_A \frac{\partial g_B}{\partial x_i} e_B\Big) \\
&= \Big(\sum_A \frac{\partial f_A}{\partial x_i} e_A\Big)\Big(\sum_B g_B e_B\Big) + \Big(\sum_A f_A e_A\Big)\Big(\sum_B \frac{\partial g_B}{\partial x_i} e_B\Big) \\
&= \frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i} \quad \text{for} \quad i = 0, 1, ..., m.
\end{aligned} \tag{2.4}$$

It follows from (2.3), (2.4) that

$$\begin{aligned}
D_x(fg) &= \sum_{i=0}^{m} e_i \Big(\frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i}\Big) \\
&= \Big(\sum_{i=0}^{m} e_i \frac{\partial f}{\partial x_i}\Big) g + \sum_{i=0}^{m} e_i f \frac{\partial g}{\partial x_i} = (D_x f) g + \sum_{i=0}^{m} e_i f \frac{\partial g}{\partial x_i}.
\end{aligned}$$

Lemma 2.2. *Let A be an open set of $\mathbb{R}^{m+1}$ and D be a domain with smooth boundary, such that $\overline{D} \Subset A,\ 0 \notin \partial D$. If $g \in C^1(A, \mathcal{A})$, then*

$$\int\limits_D e_i E(y).D_y g(y) dy - \int\limits_D D_y\big(E(y) e_i g(y)\big) dy =$$

$$\int\limits_{\partial D} \big(e_i E(y) d\sigma_y\, g(y) - d\sigma_y\, E(y) e_i g(y)\big)$$

for $i = 0, ..., m$.

Proof

a) Assume that $i = 0$ then $e_i = 1$. We will prove that

$$\int_D E.Dgdy - \int_D D(Eg)dy = \int_{\partial D} (Ed\sigma_y\, g - d\sigma_y\, Eg). \tag{2.5}$$

If $0 \notin \overline{D}$, then according to [1, Lemma 9.2], we have

$$\int_{\partial D} Ed\sigma_y\, g = \int_D (ED.g + E.Dg)dy = \int_D E.Dgdy \tag{2.6}$$

because $ED = 0$ (see [1]) and

$$\int_{\partial D} d\sigma_y\, Eg = \int_D D(Eg)dy. \tag{2.7}$$

From (2.6), (2.7) it follows (2.5).

If $0 \in D$, then we choose $R > 0$, such that $\overline{B}(0, R) \subset D$ where $B(0, R)$ is the ball in $\mathbb{R}^{m+1}$. Using (2.5) for the domain $(D \setminus B)$, we obtain

$$\int_{D\setminus B} E.Dgdy - \int_{D\setminus B} D(Eg)dy = \int_{\partial(D\setminus B)} (Ed\sigma_y\, g - d\sigma_y\, Eg). \tag{2.8}$$

Clearly, if $R \to 0^+$, then the left hand side of (2.8) tends to

$$\int_D E.Dgdy - \int_D D(Eg)dy, \text{ (see [1, Theorem 9.5])}. \tag{2.9}$$

The right hand side of (2.8) has the form

$$\int_{\partial D} (Ed\sigma_y\, g - d\sigma_y\, Eg) - \int_{\partial B(0,R)} (Ed\sigma_y\, g - d\sigma_y\, Eg). \tag{2.10}$$

Applying the method used in the proof of Theorem 9.5 (see [1]), we have

$$\lim_{R\to 0^+} \int_{\partial B(0,R)} Ed\sigma_y\, g = g(0). \tag{2.11}$$

Similarly, we can prove that

$$\lim_{R\to 0^+} \int_{\partial B(0,R)} d\sigma_y\, Eg = g(0). \tag{2.12}$$

Hence we get

$$\lim_{R\to 0^+} \int_{\partial B(0,R)} (Ed\sigma_y\, g - d\sigma_y\, Eg) = g(0) - g(0) = 0. \tag{2.13}$$

By (2.8), (2.9), (2.10) and (2.13), we obtain (2.5).

b) Using the similar method in the proof a) we are able to prove the Lemma for $i = 2, ..., m$.

Thus Lemma 2.2 is proved.

Lemma 2.3. *Let D be a bounded domain in $\mathbb{R}^{m+1}$ with smooth boundary, $f \in C^1(D, \mathcal{A})$, $x \in D$ then*

$$f(x) = \int_{\partial D} d\sigma_y\, E(y-x)f(y) - \int_{D} D_y\big(E(y-x)f(y)\big)dy. \tag{2.14}$$

Proof. According to [1, Theorem 9.5], we have

$$f(x) = \int_{\partial D} E(y-x)d\sigma_y\, f(y) - \int_{D} E(y-x).D_y f(y)dy. \tag{2.15}$$

Put $y - x = \eta,\ g(\eta) = f(x+\eta)$, then the right hand side of (2.15) reduces to

$$\int_{\partial D'} E(\eta)d\sigma_\eta\, g(\eta) - \int_{D'} E(\eta).D_\eta g(\eta)d\eta. \tag{2.16}$$

By Lemma 2.2, (2.16) is of the form

$$\int_{\partial D'} d\sigma_\eta\, E(\eta)g(\eta) - \int_{D'} D_\eta\big(E(\eta)g(\eta)\big)d\eta =$$

$$\int_{\partial D} d\sigma_y\, E(y-x)f(y) - \int_{D} D_y\big(E(y-x)f(y)\big)dy.$$

So we have (2.14).

Corollary 2.1. *Let D be a domain as in Lemma 2.3. If $f \in C_0^1(D, \mathcal{A})$, then*

$$f(x) = -\int_{D} D_y\big(E(y-x)f(y)\big)dy \quad \text{for} \quad x \in D. \tag{2.17}$$

3. The Inhomogeneous Generalized Cauchy–Riemann Systems

In this section, we consider the system

$$D_{x^{(h)}} g = f_h \quad \text{for} \quad h = 1, ..., n, \quad n \geq 2, \tag{3.1}$$

where f_h are defined in an open set $\Omega \subset \mathbb{R}^{(m+1)n}$ and take values in the Clifford algebra $\mathcal{A}$.

Note that in case $n = 1$ the system (3.1) was studied in [1].

Lemma 3.1. *Suppose that $f_h \in C^1(\Omega, \mathcal{A})$. If $g \in C^2(\Omega, \mathcal{A})$ is a solution of* (3.1)*, then*

$$\sum_{i=0}^{m} e_i E(\eta) \frac{\partial f_h}{\partial x_i^{(1)}} = \sum_{i=0}^{m} D_\eta \Big(E(\eta) e_i \frac{\partial g}{\partial x_i^{(h)}}\Big), \tag{3.2}$$

$$\sum_{i=0}^{m} e_i E(\eta) \frac{\partial f_1}{\partial x_i^{(h)}} = \sum_{i=0}^{m} e_i E(\eta).D_\eta \frac{\partial g}{\partial x_i^{(h)}} \tag{3.3}$$

for $h = 1, ..., n$, where

$$\frac{\partial f_h}{\partial x_i^{(1)}} = \frac{\partial f_h}{\partial x_i^{(1)}}(x^{(1)} + \eta, x^{(2)}, ..., x^{(n)}),$$
$$\frac{\partial f_1}{\partial x_i^{(h)}} = \frac{\partial f_1}{\partial x_i^{(h)}}(x^{(1)} + \eta, x^{(2)}, ..., x^{(n)}),$$
$$\frac{\partial g}{\partial x_i^{(h)}} = \frac{\partial g}{\partial x_i^{(h)}}(x^{(1)} + \eta, x^{(2)}, ..., x^{(n)}).$$

Proof. The right hand side of (3.2) has the form

$$\sum_{i=0}^{m} D_\eta\Big(E(\eta)e_i \frac{\partial g}{\partial x_i^{(h)}}\Big) = D_\eta\Big(E(\eta)\sum_{i=0}^{m} e_i \frac{\partial g}{\partial x_i^{(h)}}\Big)$$
$$= D_\eta(E(\eta)D_{x^{(h)}}g) = D_\eta(E(\eta)f_h) \quad \text{because} \quad D_{x^{(h)}}g = f_h. \tag{3.4}$$

By Lemma 2.1, (3.4) reduces to

$$D_\eta(E(\eta)f_h) = (D_\eta E(\eta))f_h + \sum_{i=0}^{m} e_i E(\eta)\frac{\partial f_h}{\partial \eta_i} \quad \text{where} \quad \eta = (\eta_0, ..., \eta_m)$$
$$= 0 + \sum_{i=0}^{m} e_i E(\eta)\frac{\partial f_h}{\partial x_i^{(1)}}$$
$$= \sum_{i=0}^{m} e_i E(\eta)\frac{\partial f_h}{\partial x_i^{(1)}} \quad \text{since} \quad E(\eta) \quad \text{is regular (see [1]).}$$

Thus (3.2) is proved.

On the other hand, we get

$$D_\eta \frac{\partial g}{\partial x_i^{(h)}} = D_{x^{(1)}}\frac{\partial g}{\partial x_i^{(h)}} = \sum_{j=0}^{m} e_j \frac{\partial}{\partial x_j^{(1)}}\Big(\frac{\partial g}{\partial x_i^{(h)}}\Big)$$
$$= \sum_{j=0}^{m} e_j \frac{\partial}{\partial x_j^{(1)}}\Big(\sum_B e_B \frac{\partial g_B}{\partial x_i^{(h)}}\Big) = \sum_{j=0}^{m}\sum_B e_j e_B \frac{\partial^2 g_B}{\partial x_j^{(1)}\partial x_i^{(h)}}$$
$$= \sum_{j=0}^{m}\sum_B e_j e_B \frac{\partial}{\partial x_i^{(h)}}\Big(\frac{\partial g_B}{\partial x_j^{(1)}}\Big) = \frac{\partial}{\partial x_i^{(h)}}\Big(\sum_{j=0}^{m} e_j \frac{\partial}{\partial x_j^{(1)}}\big(\sum_B e_B g_B\big)\Big)$$
$$= \frac{\partial}{\partial x_i^{(h)}}\Big(\sum_{j=0}^{m} e_j \frac{\partial g}{\partial x_j^{(1)}}\Big) = \frac{\partial}{\partial x_i^{(h)}}(D_{x^{(1)}}g) = \frac{\partial f_1}{\partial x_i^{(h)}}. \tag{3.5}$$

It follows from (3.5) that

$$e_i E(\eta)\frac{\partial f_1}{\partial x_i^{(h)}} = e_i E(\eta).D_\eta \frac{\partial g}{\partial x_i^{(h)}} \quad \text{for} \quad i = 0, ..., m. \tag{3.6}$$

Hence we obtain (3.3).

Theorem. *Suppose that $f_1, ..., f_n \in C_0^k(\mathbb{R}^{(m+1)n}, \mathcal{A})$, $k \geq 2$, then the system* (3.1) *has at least a solution $g \in C_0^k(\mathbb{R}^{(m+1)n}, \mathcal{A})$ if and only if the following condition is satisfied*

$$\int\limits_{\mathbb{R}^{m+1}} \sum_{i=0}^{m} e_i E(\eta) \frac{\partial f_h}{\partial x_i^{(1)}} d\eta = \int\limits_{\mathbb{R}^{m+1}} \sum_{i=0}^{m} e_i E(\eta) \frac{\partial f_1}{\partial x_i^{(h)}} d\eta \tag{3.7}$$

for $h = 2, ..., n$ and for all $(x^{(1)}, ..., x^{(n)}) \in \mathbb{R}^{(m+1)n}$, where $\frac{\partial f_h}{\partial x_i^{(1)}}$ and $\frac{\partial f_1}{\partial x_i^{(h)}}$ are defined as in Lemma 3.1. *When the conditions of the Theorem are satisfied then the solution $g \in C_0^k(\mathbb{R}^{(m+1)n}, \mathcal{A})$ is uniquely determined.*

Proof

1. Necessity

Suppose that the system (3.1) has a solution $g \in C_0^k(\mathbb{R}^{(m+1)n}, \mathcal{A})$, we will prove that (3.7) is valid.

According to Lemma 3.1, we get

$$\sum_{i=0}^{m} e_i E(\eta) \frac{\partial f_h}{\partial x_i^{(1)}} = \sum_{i=0}^{m} D_\eta \Big(E(\eta) e_i \frac{\partial g}{\partial x_i^{(h)}} \Big), \tag{3.8}$$

$$\sum_{i=0}^{m} e_i E(\eta) \frac{\partial f_1}{\partial x_i^{(h)}} = \sum_{i=0}^{m} e_i E(\eta). D_\eta \frac{\partial g}{\partial x_i^{(h)}}. \tag{3.9}$$

Since $g \in C_0^k(\mathbb{R}^{(m+1)n}, \mathcal{A})$, $\frac{\partial g}{\partial x_i^{(h)}}$ also have such a property.

It follows from Lemma 2.2 that

$$\int\limits_{\mathbb{R}^{m+1}} \sum_{i=0}^{m} D_\eta \Big(E(\eta) e_i \frac{\partial g}{\partial x_i^{(h)}} \Big) d\eta = \int\limits_{\mathbb{R}^{m+1}} \sum_{i=0}^{m} e_i E(\eta). D_\eta \frac{\partial g}{\partial x_i^{(h)}} d\eta. \tag{3.10}$$

By (3.8), (3.9) and (3.10) we have (3.7).

2. Sufficiency

Assume that (3.7) is satisfied. Set

$$\begin{aligned} g(x^{(1)}, ..., x^{(n)}) &= - \int\limits_{\mathbb{R}^{m+1}} E(y - x^{(1)}) f_1(y, x^{(2)}, ..., x^{(n)}) dy \\ &= - \int\limits_{\mathbb{R}^{m+1}} E(\eta) f_1(x^{(1)} + \eta, x^{(2)}, ..., x^{(n)}) d\eta, \end{aligned} \tag{3.11}$$

then

$$D_{x^{(h)}} g(x^{(1)}, ..., x^{(n)}) = - \int\limits_{\mathbb{R}^{m+1}} D_{x^{(h)}} E(\eta) f_1(x^{(1)} + \eta, x^{(2)}, ..., x^{(n)}) d\eta. \tag{3.12}$$

Using Lemma 2.1 we see that the right hand side of (3.12) has the form

$$-\int_{\mathbb{R}^{m+1}}\Big[\big(D_{x^{(h)}}E(\eta)\big)f_1(x^{(1)}+\eta,x^{(2)},...,x^{(n)}) + \sum_{i=0}^{m} e_iE(\eta)\frac{\partial f_1}{\partial x_i^{(h)}}(x^{(1)}+\eta,x^{(2)},...,x^{(n)})\Big]d\eta. \tag{3.13}$$

i) If $h=1$, then $D_{x^{(h)}}E(\eta)=0$, because $E(\eta)$ is regular.
ii) If $h\geq 2$, then $D_{x^{(h)}}E(\eta)=0$, because $E(\eta)$ is independent of $x^{(h)}$.

Hence (3.13) reduces to

$$-\int_{\mathbb{R}^{m+1}}\sum_{i=0}^{m} e_iE(\eta)\frac{\partial f_1}{\partial x_i^{(h)}}(x^{(1)}+\eta,x^{(2)},...,x^{(n)})d\eta. \tag{3.14}$$

From (3.7) and Corollary 2.1, (3.11) becomes

$$\begin{aligned}&-\int_{\mathbb{R}^{m+1}}\sum_{i=0}^{m} e_iE(\eta)\frac{\partial f_h}{\partial x_i^{(1)}}(x^{(1)}+\eta,x^{(2)},...,x^{(n)})d\eta = \\ &-\int_{\mathbb{R}^{m+1}} D_y\big(E(y-x^{(1)})f_h(y,x^{(2)},...,x^{(n)})\big)dy = f_h(x^{(1)},...,x^{(n)})\end{aligned} \tag{3.15}$$

for $h=1,...,n$.

According to (3.12)–(3.15), g defined by (3.11), is a solution of (3.1).

Finally, we will show that g has a compact support.

It is easy to see that $g(x^{(1)},...,x^{(n)})=0$ for all $x^{(1)}\in\mathbb{R}^{m+1}$ if $\Big(\sum_{i=2}^{n}|x^{(i)}|^2\Big)^{1/2}$ is large enough. So $g=0$ in the unbounded connected component of $\mathbb{R}^{(m+1)n}$. On the other hand, f_h have compact supports, then g is multi-regular outside of a compact disc C of $\mathbb{R}^{(m+1)n}$.

From the Uniqueness theorem for multi-regular functions (see Remark 1.1), it follows that $g=0$ outside C. Hence g has a compact support.

Now we are able to prove that the solution $g\in C_0^k(\mathbb{R}^{(m+1)n},\mathcal{A})$ of system (3.1) defined by (3.11), is uniquely determined.

In fact, suppose that g and g_1 are solutions with compact supports of system (3.1). Then we have

$$D_{x^{(h)}}(g-g_1)=f_h-f_h=0 \quad \text{in } \mathbb{R}^{(m+1)n} \text{ for all } h=1,...,n.$$

Hence $\varphi=g-g_1$ is a multi-regular function in $\mathbb{R}^{(m+1)n}$. Since g and g_1 have compact supports, then φ has also a compact support. According to the Uniqueness theorem for multi-regular functions (Remark 1.1), it follows that $\varphi=0$ in the whole $\mathbb{R}^{(m+1)n}$.

Thus $g=g_1$ in $\mathbb{R}^{(m+1)n}$.

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