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Survey

Tame Topology and Tarski-type Systems

Ta Lê Loi

Department of Mathematics, University of Dalat, Dalat, Vietnam

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Abstract. This note is an overview of the theory of o-minimal structures on the real field, which is a generalization of semi-algebraic and subanalytic geometry.

1. Introduction

In Real Algebraic and Analytic Geometry the following classes of sets and their geometries are considered:
(1) The class of semialgebraic sets (Whitney-Lojasiewicz, in the 50's) (see [1]).
(2) The class of semianalytic sets (Lojasiewicz, in the 60's) (see [22]).
(3) The class of subanalytic sets (Gabrielov-Hironaka-Hardt and Krakovian school, in the 70's) (see [2, 10, 23]).

These classes of sets have many nice properties. Semialgebraic and subanalytic sets form the so-called *Tarski-type systems*, that is the corresponding class is closed under boolean operators and under proper projections. In particular, these classes have the finiteness property: each set in these classes has locally only finite number of connected components and each of the components also belongs to the corresponding class.

In some problems, we have to treat functions like x^α or $\exp(-1/x)$, where $x > 0$ and α is a irrational number, which are not subanalytic at 0. Naturally, it is required an extension of classes mentioned above. According to van den Dries [4], the finiteness is the most remarkable property in the sense that if a Tarski-type system has this property, it would preserve many nice properties of semi and subanalytic sets. Van den Dries, Knight, Pillay and Steinhorn gave the name *o-minimal structures* for such systems and developed the general theory [4, 12, 25] (c.f. [26, 27]). Khovanskii's results on Fewnomials [11], and a notable theorem of Wilkie on model completeness [28] confirm the o-minimality of the real exponential field. In recent years, o-minimality of many remakable structures have been proved and many interesting results have been established in the theory of o-minimal structures on the real field.

In this note, we will make an overview of the theory of o-minimal structures on the real field. The definition and some examples of o-minimal structures are given in Sec. 2. In Sec. 3 we list some important properties of o-minimal structures. The proofs can be found in papers or books given in the references. In the last section we sketch the idea of constructing an analytic-geometric category corresponding to an o-minimal structures.

2. O-minimal Structures on $(\mathbb{R}, +, \cdot)$

2.1. Motivation. Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a class of real-valued functions on spaces $\mathbb{R}^n$, $n \in \mathbb{N}$. Similar to semialgebraic or subanalytic sets, it is natural to construct a class of subsets of $\mathbb{R}^n$, $n \in \mathbb{N}$, as follows.

First consider basic sets of the forms

$$\{x \in \mathbb{R}^n : f(x) > 0\}, \quad \text{where} \quad f \in \mathcal{F}_n,\ n \in \mathbb{N}.$$

Then starting from these basic sets we create new sets by taking: finite unions, finite intersections, complements, Cartersian products, and linear projections (or proper projections) onto smaller dimensional Euclidean spaces. Repeating these operators with the new sets that arise, we get a class of subsets of $\mathbb{R}^n, n \in \mathbb{N}$, which is closed under usual topological operators (e.g. taking closure, interior, boundary, ...).

We are interested in the case that the new sets are not so complicated and pathological as Cantor sets, Borel sets, nonmeasurable sets..., since it promises a “tame topology” for the class of sets that we constructed. The corresponding category of spaces and maps between them may yield a rich algebraic-analytic-topological structures.

Definition 1. *A structure on the real field $(\mathbb{R}, +, \cdot)$ is a sequence $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ such that the following conditions are satisfied for all $n \in \mathbb{N}$:*

(D1) *$\mathcal{D}_n$ is a boolean algebra of subsets of $\mathbb{R}^n$.*

(D2) *If $A \in \mathcal{D}_n$, then $A \times \mathbb{R}$ and $\mathbb{R} \times A \in \mathcal{D}_{n+1}$.*

(D3) *If $A \in \mathcal{D}_{n+1}$, then $\pi(A) \in \mathcal{D}_n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates.*

(D4) *$\mathcal{D}_n$ contains $\{x \in \mathbb{R}^n : P(x) = 0\}$ for every polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$.*

Structure $\mathcal{D}$ is called to be *o-minimal* if

(D5) *Each set in $\mathcal{D}_1$ is a finite union of intervals and points.*

A set belonging to $\mathcal{D}$ is called definable (in that structure). Definable maps in structure $\mathcal{D}$ are maps whose graphs are definable sets in $\mathcal{D}$.

Examples. Given a collection of real-valued functions $\mathcal{F}$, the smallest structure on $(\mathbb{R}, +, \cdot)$ containing the graphs of all $f \in \mathcal{F}$ is denoted by $(\mathbb{R}, +, \cdot, \mathcal{F})$.

1. Let $\mathbb{R}_{\text{alg}}$ be the smallest structure on $(\mathbb{R}, \cdot, +)$. By Tarski-Seidenberg's Theorem a subset $X \subset \mathbb{R}^n$ is definable in $\mathbb{R}_{\text{alg}}$ if and only if X is semialgebraic. Obviously, $\mathbb{R}_{\text{alg}}$ is o-minimal.

2. Let $\mathbb{R}_{\text{an}} = (\mathbb{R}, +, \cdot, \mathcal{A})$, where $\mathcal{A}$ is the class of all restricted analytic functions on $[-1, 1]^n$ $(n \in \mathbb{N})$. Definable sets in $\mathbb{R}_{\text{an}}$ are finitely subanalytic sets (see [5]), i.e $X \subset \mathbb{R}^n$ is definable in $\mathbb{R}_{\text{an}}$ if and only if X is subanalytic in the projective space $\mathbf{P}^n(\mathbb{R})$, where we identify $\mathbb{R}^n$ with an open set of $\mathbf{P}^n(\mathbb{R})$ via $(x_1, \ldots, x_n) \mapsto (1 : x_1 : \cdots : x_n)$. By Grabrielov's Theorem on the complement and a Lojasiewicz result on connected components of semianalytic sets (see [2, 22, 23]) $\mathbb{R}_{\text{an}}$ is o-minimal.

3. Let $\mathbb{R}_{\exp} = (\mathbb{R}, +, \cdot, \exp)$. Wilkie [28] proved that $\mathbb{R}_{\exp}$ is model complete, as a direct consequence of this theorem each definable sets in $\mathbb{R}_{\exp}$ is the image of the zero set of a function in $\mathbb{R}[x_1, \ldots, x_N, \exp(x_1), \ldots, \exp(x_N)]$, for some $N \in \mathbb{N}$ under a natural projection (see [16]). Then by a Khovanskii result on fewnomials [11], $\mathbb{R}_{\exp}$ is an o-minimal structure. An analytic proof of Wilkie's theorem is given in [21]. Note that $x^\alpha, \exp\Big(-\frac{1}{x}\Big)$ $(x > 0$ and α is irrational) are definable functions in $\mathbb{R}_{\exp}$ but not subanalytic at 0.

4. Let $\mathbb{R}_{\text{an, exp}} = (\mathbb{R}, +, \cdot, \mathcal{A}, \exp)$, where $\mathcal{A}$ as in 3.2. Extending Wilkie's method, van den Dries and Miller [6] proved that $\mathbb{R}_{\text{an, exp}}$ is also o-minimal. (see also [21] for an analytic proof)

5. Let $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ be a *Pfaffian chain* , i.e. $f_1, \ldots, f_k$ are smooth functions and there exist polynomials $P_{ij} \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_i]$ such that for all $x \in \mathbb{R}^n$

$$\frac{\partial f_i}{\partial x_j}(x) = P_{ij}(x, f_1(x), \ldots, f_i(x)) \quad (i = 1, \ldots, k : j = 1, \ldots, n).$$

Let $\mathcal{P} = \mathcal{P}(f_1, \ldots, f_k)$ be the class of all functions of the form $f(x) = Q(x, f_1(x), \ldots, f_k(x))$, where $Q \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_k]$, (they are called *Pfaffian functions*). Then, by Khovanskii's Theory [11] and by a result of Wilkie [29], $(\mathbb{R}, +, \cdot, \mathcal{P})$ is o-minimal.

3. Some Properties of o-Minimal Structures

Throughout this section $\mathcal{D}$ denotes an o-minimal structure on $(\mathbb{R}, +, \cdot)$. "Definable" means definable in $\mathcal{D}$.

We now list some important results of $\mathcal{D}$.

3.1. Elementary Properties

(i) *The closure, the interior, and the boundary of a definable set are definable.*
(ii) *Compositions of definable maps are definable.*
(iii) *Images and inverse images of definable sets under definable maps are definable.*

In particular, these properties imply that if $f : S \to \mathbb{R}$ is a definable function then the sets $C^p(f) = \{x \in S : f \text{ is of class } C^p \text{ at } x\}$, $p \in \mathbb{N}$, are definable. Note that these properties hold for any structure not necessary o-minimal.

3.2. Cell Decomposition

Let p be a positive integer.

Definition 2. *C^p cells in $\mathbb{R}^n$ are connected C^p-submanifolds of $\mathbb{R}^n$ belonging to $\mathcal{D}$ which are defined by induction on n as follows:*

- The C^p cells in $\mathbb{R}$ are points or open intervals.

- If $C \subset \mathbb{R}^n$ is a C^p cell and $f, g : C \to \mathbb{R}$ are definable functions of class C^p such that $f < g$, then the sets: $C \times \mathbb{R}$, $\Gamma(f) = \{(x,t) : t = f(x)\}$, $(f,g) = \{(x,t) : f(x) < t < g(x)\}$, $(-\infty, f) = \{(x,t) : t < f(x)\}$ and $(f, +\infty) = \{(x,t) : f(x) < t\}$ are C^p cells in $\mathbb{R}^{n+1}$.

A C^p decomposition of $\mathbb{R}^n$ is defined by induction on n:

- A C^p decomposition of $\mathbb{R}$ is a finite collection of intervals and points $\{(-\infty, a_1), \dots, (a_k, +\infty), \{a_1\}, \dots, \{a_k\}\}$, where $a_1 < \cdots < a_k$, $k \in \mathbb{N}$.

- A C^p decomposition of $\mathbb{R}^{n+1}$ is a finite partition of $\mathbb{R}^{n+1}$ into C^p cells C, such that the collection of all the projections $\pi(C)$ is a C^p decomposition of $\mathbb{R}^n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates.

We say that a decomposition *compatible with* a class $\mathcal{A}$ of subsets of $\mathbb{R}^n$, if each $S \in \mathcal{A}$ is a union of some cells of the decomposition.

Theorem 1. [5]

(i) *For $S_1, \dots, S_k \in \mathcal{D}_n$, there exists a C^p decomposition of $\mathbb{R}^n$ compatible with $\{S_1, \dots, S_k\}$.*

(ii) *For each definable function $f : S \to \mathbb{R}$, $S \subset \mathbb{R}^n$, there exists a C^p decomposition of $\mathbb{R}^n$ compatible with S such that for each cell $C \subset S$ of the decomposition the restriction $f|_C$ is of class C^p.*

Note: The structures given in Examples 1 - 4 admit analytic decomposition, i.e. the theorem still holds true if we replace "C^p" by "analytic".

The theorem implies

Theorem 2. (on components). *Every definable set has only finitely many connected components and each component is also definable.*

3.3. Definable Selection [5, 17]

Let $S \subset \mathbb{R}^m \times \mathbb{R}^n$ be a definable set and let $\pi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ be the projection on the first m coordinates. Then there exists a definable map $\rho : \pi(S) \to \mathbb{R}^m \times \mathbb{R}^n$ such that $\pi(\rho(x)) = x$, for all $x \in \pi(S)$.

In particular, if $a \in \overline{S}$ is a nonisolated point of S, then there exists a C^p definable curve $\gamma : (0,1) \to S$ such that $\lim\limits_{t \to 0^+} \gamma(t) = a$.

Note: For sets definable in $\mathbb{R}_{\text{alg}}$ or $\mathbb{R}_{\text{an}}$, by Puiseux lemma, γ can be chosen to be analytic on $(-1, 1)$. For sets in structure $\mathbb{R}_{\exp}$ or $\mathbb{R}_{\text{an, exp}}$, the theorem holds true for analytic curve γ; but, in general, it cannot be analytically extended to 0 (e.g. $S = \{(x, y) : x > 0, y = \exp(-1/x)\}$).

3.4. Dimension

Let $S \subset \mathbb{R}^n$ be a definable set. The dimension of S is defined by

$$\dim S = \{\dim C : C \text{ is a } C^p\text{-submanifold of } \mathbb{R}^n \text{ containing in } S\}.$$

Proposition. [5]
(i) $\dim(\overline{S} \setminus S) < \dim S$.
(ii) $\dim f(S) \leq \dim S$, *for every* $\mathcal{D}$*-map* $f : S \to \mathbb{R}^m$.

3.5. Stratifications

Whitney and Verdier Conditions. Let Γ, Γ' be C^1 submanifolds of $\mathbb{R}^n$ such that $\Gamma \subset \overline{\Gamma}' \setminus \Gamma'$. Let $T_y\Gamma$ denotes the tangent space of Γ at y, and $\delta(T, T') = \sup_{v \in T, \|v\|=1} d(v, T')$ is the distance of vector subspaces of $\mathbb{R}^n$. Let y_0 be a point of Γ.

We say that the pair (Γ, Γ') satisfies the Whitney condition at y_0 if the following holds:
(b) If $x \in \Gamma' \cap U, y \in \Gamma \cap U$, $\|x - y\| \to 0$, then $\delta(T_y\Gamma, T_x\Gamma') \to 0$.

We say that the pair (Γ, Γ') satisfies the Verdier condition at y_0 if the following holds:
(w) There exists a constant $C > 0$ and a neighborhood U of y_0 in $\mathbb{R}^n$ such that
$$\delta(T_y\Gamma, T_x\Gamma') \leq C\|x - y\| \quad \text{for all} \quad x \in \Gamma' \cap U, y \in \Gamma \cap U,$$

Definition 3. *A* C^p *stratification of* $\mathbb{R}^n$ *is a partition* $\mathcal{S}$ *of* $\mathbb{R}^n$ *into finitely many subsets, called strata, such that:*
(S1) *Each stratum is a connected* C^p *submanifold of* $\mathbb{R}^n$ *and also definable set.*
(S2) *For every* $\Gamma \in \mathcal{S}$, $\overline{\Gamma} \setminus \Gamma$ *is a union of some of the strata.*

A C^p *Whitney stratification* (resp. *Verdier stratification*) is a C^p stratification $\mathcal{S}$ such that for all $\Gamma, \Gamma' \in \mathcal{S}$, if $\Gamma \subset \overline{\Gamma'} \setminus \Gamma'$, then (Γ, Γ') satisfies the condition (b) (resp. (w)) at each point of Γ.

Theorem 3. [19] *Let* $S_1, \ldots, S_k$ *be definable sets in* $\mathbb{R}^n$. *Then there exists a* C^p *Verdier stratification of* $\mathbb{R}^n$ *compatible with* $\{S_1, \ldots, S_k\}$.

In o-minimal structures the Verdier condition (w) implies the Whitney condition (b) (see [19]), so we have

Theorem 4. [7] (c.f. [17, 27, 19]) *Let* $S_1, \ldots, S_k$ *be definable sets in* $\mathbb{R}^n$. *Then there exists a* C^p *Whitney stratification of* $\mathbb{R}^n$ *compatible with* $\{S_1, \ldots, S_k\}$.

Definition 4. *Let* $S \subset \mathbb{R}^n$ *be a* $\mathcal{D}$*-sets and* $f : S \to \mathbb{R}^m$ *be a definable map. A* C^p *Whitney stratification of* f *is a pair* $(\mathcal{S}, \mathcal{T})$*, where* $\mathcal{S}$ *and* $\mathcal{T}$ *are* C^p *Whitney stratifications of* $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *respectively, such that* $\mathcal{S}$ *is compatible with* S *and* *for each $\Gamma \in \mathcal{S}$ if $\Gamma \subset S$, then there exists $\Phi \in \mathcal{T}$ such that $f|_\Gamma : \Gamma \to \Phi$ is a C^p submersion.*

Theorem 5. [20] *Let $f : S \to \mathbb{R}^m$ be a definable map. Then there exists a C^p Whitney stratification of f.*

Thom stratifications. Let $f : S \to \mathbb{R}$ be a continuous definable function. Let $\mathcal{S}$ be a stratification of f. For each $x \in \Gamma$, $T_{x,f}$ denotes the tangent space of the level of $f|_\Gamma$ at x, i.e. $T_{x,f} = \ker d(f|_\Gamma)(x)$.

Let $\Gamma, \Gamma' \in \mathcal{S}$ with $\Gamma \subset \overline{\Gamma'} \setminus \Gamma'$. We say that the pair (Γ, Γ') satisfies the *Thom condition* (a_f) at $y_0 \in \Gamma$ if and only if the following holds:
(a_f) for every sequence (x_k) in Γ', converging to y_0, we have

$$\delta(T_{y_0,f}, T_{x_k,f}) \longrightarrow 0.$$

We say that (Γ, Γ') satisfies the *strict Thom condition* (w_f) at y_0 if:
(w_f) there exists a constant $C > 0$ and a neighborhood U of y_0 in $\mathbb{R}^n$, such that

$$\delta(T_{y,f}, T_{x,f}) \leq C\|x - y\| \quad \text{for all} \quad x \in \Gamma' \cap U, \quad y \in \Gamma \cap U.$$

Theorem 6. [18] *There exists a C^p stratification of f satisfying the Thom condition (a_f) at every point of the strata.*

In general, definable functions cannot be stratified to satisfy the condition (w_f). For example, the function $f : (a, b) \times (0, +\infty) \to \mathbb{R}$ defined by $f(x, y) = y^x$ $(0 < a < b)$.

However, in polynomially bounded structures (see the definition 8) we have

Theorem 7. [19] *Suppose that $\mathcal{D}$ is polynomially bounded. Then there exists a C^p stratification of f satisfying the condition (w_f) at each point of the strata.*

3.6. Trivializations

Definition 5. *Let $f : S \to \mathbb{R}^m$, $S \subset \mathbb{R}^n$, be a definable map. A definable trivialization of f is a pair (F, λ) consisting of a definable set F and a definable map $\lambda : S \to F$, such that $(f, \lambda) : S \to f(S) \times F$ is a homeomorphism.*

The following is a version of Hardt's theorem on trivialization [9].

Theorem 8. [5] *Let $f : S \to \mathbb{R}^m$ be a continuous definable map. Then there is a partition of $f(S) = C_1 \bigcup \cdots \bigcup C_k$ into definable sets such that $f|_{f^{-1}(C_i)}$ can be definable trivialized for $i = 1, \ldots, k$.*

3.7. Triangulations

Whitney stratified spaces can be triangulated. In o-minimal setting we have

Theorem 9. [5] *Let $S_1, \ldots, S_k \subset \mathbb{R}^n$ be a definable sets. Then $S = \bigcup_{i=1}^{k} S_i$ admits a triangulation compartible with $\{S_1, \ldots, S_k\}$, i.e there exist a simplicial* *complex K and a homeomorphism $h : |K| \to S$ belonging to $\mathcal{D}$ such that each S_i is a union of some elements of $\{h(\sigma) : \sigma \in K\}$.*

Theorem 10. [3, 27] *Let $f : S \to \mathbb{R}$ be a definable function. Then there exist a simplicial complex K and a definable homeomorphism $h : |K| \to S$ such that $f \circ h$ is linear on each simplex of K.*

Corollary. [3] (c.f. [20]) *Let $f : S \times T \to \mathbb{R}, (x,t) \mapsto f_t(x)$, be a definable family of functions. Then there exists finite partition $T = \cup_{i=1}^k T_i$ into C^p definable manifolds, such that for t and t' in the same T_i, f_t and f'_t are topologically equivalent, that is there exist definable homeomorphisms $h : S \to S$, and $\lambda : \mathbb{R} \to \mathbb{R}$, such that $f_t \circ f = \lambda \circ f_{t'}$*

The corollary is an extension of [8], where Fukuda proved that the number of topological types of polynomial functions on $\mathbb{R}^n$ of degree $\leq d$ is finite.

3.8. Asymptotic Behaviours

Monotonicity [5, 12, 25] *Let $f : (a,b) \to \mathbb{R}$ be a definable function. Then there exists a partition $a = a_0 < a_1 < \cdots < a_{k+1} = b$ such that $f|(a_i, a_{i+1})$ is C^p, and either constant or strictly monotone, for $i = 1, \ldots, k$.*

Note: From Monotonicity, the germs at $+\infty$ of $\mathcal{D}$-functions on $\mathbb{R}$ forms a *Hardy field* , i.e a set of germs at $+\infty$ of real-valued functions on neighborhoods of $+\infty$ that is closed under differentiation and that form a field with usual addition and multiplication of germs.

Definition 6. *$\mathcal{D}$ is polynomially bounded if for every definable function $f : \mathbb{R} \to \mathbb{R}$, there exists $n \in \mathbb{N}$ such that*

$$|f(t)| \;\leq\; t^n, \;\; \textit{for all sufficiently large} \;\; t.$$

$\mathcal{D}$ is exponentially bounded if for every definable function $f : \mathbb{R} \to \mathbb{R}$, there exists $n \in \mathbb{N}$ such that

$$|f(t)| \;\leq\; \exp_n(t), \quad \textit{for all sufficiently large} \;\; t,$$

where $\exp_0(t) = t$ *and* $\exp_{m+1}(t) = \exp(\exp_m(t)),\; m \in \mathbb{N}$.

Note: $\mathbb{R}_{\mathrm{alg}}$, $\mathbb{R}_{\mathrm{an}}$ are polynomially bounded; $\mathbb{R}_{\exp}$, $\mathbb{R}_{\mathrm{an,\,exp}}$ are exponentially bounded (see [6]).

Uniform bounds. *Let $f : X \times \mathbb{R} \to \mathbb{R}$ be a definable function. Then there exists a definable function $\varphi : \mathbb{R} \to \mathbb{R}$ such that*

$$|f(x,t)| \;\leq\; \varphi(t) \;, \quad \textit{for sufficiently large } (\textit{depending on } x) \; t.$$

This is proved by the author in a letter to C. Miller (see [7]).

Piecewise Uniform Asymptotics [24]. Suppose that $\mathcal{D}$ is polynomially bounded. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be definable. Then there exists a partition $\{X_1, \ldots, X_l\}$ of

$\mathbb{R}^n$ into cells such that for each $i \in \{1, \ldots, l\}$ either $f(x,t) = 0$ for all $x \in X_i$ and sufficiently small (depending on x) positive t, or there exist $r_i \in \mathbb{R}$ and a continuous definable function $c_i : X_i \to \mathbb{R} \setminus 0$ such that $f(x,t) = c_i(x)t^{r_i} + o(t^{r_i})$ as $t \to 0^+$, for all $x \in X_i$.

3.9. Lojasiewicz-type Inequalities.

Let Φ^p denote the set of all odd, strictly increasing C^p definable bijection from $\mathbb{R}$ onto $\mathbb{R}$ and p-flat at 0.

Theorem 11. [6] (see also [15, 16, 14])

(i) *Let $f, g : S \to \mathbb{R}$ be continuous definable functions on closed subset S in $\mathbb{R}^n$. Suppose that $f^{-1}(0) \subset g^{-1}(0)$. Then there exist $p \in \mathbb{N}, \varphi \in \Phi^p$ and a continuous function h on S such that $\varphi(g) = hf$.*
In particular, there exist $\varphi, \varphi' \in \Phi^p$ such that

$$|f(x)| \geq \varphi(|g(x)|), \ \forall x \in S,$$
$$|f(x)| \geq \varphi'(\text{dist } (x, f^{-1}(0)), \ \forall x \in S,$$

(ii) *Let X, Y be closed definable sets in $\mathbb{R}^n$. Then there exist $p \in \mathbb{N}, \varphi \in \Phi^p$ such that*

$$\text{dist } (x, X) + \text{dist } (x, Y) \geq \varphi(\text{dist } (x, X \cap Y)), \quad \forall x \in \mathbb{R}^n.$$

(iii) *Let $f : U \to \mathbb{R}$ be a C^1 definable function on an open subset U of $\mathbb{R}^n$. Suppose that $0 \in \overline{U}$ and $\lim_{x \to 0} f(x) = 0$. Then there exist $p \in \mathbb{N}, \varphi \in \Phi^p$ such that*

$$|\text{grad } f(x)| \ \geq \ \varphi^{-1}(|f(x)|), \quad \text{for} \quad x \in U \ \text{ close to } 0.$$

Note: If $\mathcal{D}$ is polynomially bounded then φ is of the form $\varphi(t) = \text{sign } (t)t^\alpha, \alpha > 0$. If $\mathcal{D}$ is exponentially bounded then $\varphi(t) = \text{sign } (t)\dfrac{1}{\exp_m(1/|t|)}$, for some $m \in \mathbb{N}$.

3.10. Metric Property

(i) Let $S \in \mathcal{D}_n$ be connected and compact. Then there exist a continuous definable function $\gamma : S^2 \times [0,1] \to S$ and $\varphi \in \Phi^p$, such that for every $x, y \in S$

$$\text{length}(\gamma_{x,y}) \ \leq \ \varphi^{-1}(\|x - y\|),$$

where $\gamma_{x,y}(t) = \gamma(x,y,t), t \in [0,1]$, being a path from x to y in S.

(ii) Let $\mathcal{A}$ be a finite collection of definable sets in $\mathbb{R}^n$. Then there exists a stratification $\mathcal{S}$ of $\mathbb{R}^n$ compatible with $\mathcal{A}$, such that for all $\Gamma \in \mathcal{S}$, and $x, y \in \Gamma$ there exists a definable path $\gamma_{x,y}$ from x to y in Γ with $\text{length}(\gamma_{x,y}) \leq C\|x - y\|$, where C is a constant depending only on n.

(i) is given in [6]. (ii) can be proved by the same argument of [13].

4. Globalization

The notions of definable sets and their properties can be globalized in a natural way to arbitary analytic manifolds. Here we sketch the idea due to van den Dries and Miller [7].

4.1. Analytic-Geometric Categories

We say that an analytic-geometric category $\mathcal{C}$ is given if each manifold M (= real analytic, Hausdorff manifold with a countable basis for its topology) is equipped with a collection $\mathcal{C}(M)$ of subsets of M such that the following conditions are satisfied for all manifolds M and N:

(AG1) $\mathcal{C}(M)$ is a boolean algebra of subsets of M, with $M \in \mathcal{C}(M)$.
(AG2) If $A \in \mathcal{C}(M)$, then $A \times \mathbb{R} \in \mathcal{C}(M \times \mathbb{R})$.
(AG3) If $f : M \to N$ is a proper analytic map and $A \in \mathcal{C}(M)$, then $f(A) \in \mathcal{C}(N)$.
(AG4) If $A \in \mathcal{C}(M)$ and $(U_i)_{i \in I}$ is an open covering of M, then $A \in \mathcal{C}(M)$ if and only if $A \cap U_i \in \mathcal{C}(U_i)$, for all $i \in I$.
(AG5) Every bounded set in $\mathcal{C}(\mathbb{R})$ has finite boundary.

It is proved in [7] that $\mathcal{C}$ is a category with its objects being pairs (A, M), where M is a manifold and $A \in \mathcal{C}(M)$, and its morphisms $(A, M) \to (B, N)$ being maps $f : A \to B$ whose graphs belong to $\mathcal{C}(M \times N)$.

4.2. Analytic-Geometric Categories Corresponding to o-minimal Structures on $\mathbb{R}_{\text{an}}$

Let $\mathcal{D}$ be an o-minimal structure on $\mathbb{R}_{\text{an}}$. One can construct an analytic-geometry category $\mathcal{C}$ by defining the collection $\mathcal{C}(M)$ in a manifold M to be those sets $A \subset M$ such that for each $x \in M$, there exist an open neighborhood U of x, an open set $V \subset \mathbb{R}^n$, and an analytic homeomorphism $h : U \to V$ such that $h(A \cap U) \in \mathcal{D}_n$.

From the definition it follows that the category $\mathcal{C}_{\text{an}}$ of subanalytic sets and continuous subanlytic maps is the smallest analytic-geometric category corresponding to the structure $\mathbb{R}_{\text{an}}$.

4.3. Properties of Analytic-Geometry Categories

Results in Sec. 2 can be translated to the setting of analytic-geometric categories. Perhaps the only thing one has to change is to replace "finite" by "locally finite".

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