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Short communication

dd-Sequences and Partial Euler-Poincaré Characteristics of Koszul Complex

Nguyen Tu Cuong[1] **and Doan Trung Cuong**[2]

[1] *Institute of Mathematics, 18 Hoang Quoc Viet Road, 10307 Hanoi, Vietnam*
[2] *Department of Mathematics, Hanoi Unversity of Science,*
334 Nguyen Trai, Hanoi, Vietnam

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1. Introduction

Let $(R, \mathfrak{m})$ be a local commutative Noetherian ring with maximal ideal $\mathfrak{m}$ and M a finitely generated R-module. It is well-known that the theory of d-sequences [9], especially the theory of unconditioned strong d-sequences [8], plays a very important role for the studying of Buchsbaum and generalized Cohen-Macaulay modules. Recall that a sequence $\underline{x} = x_1, \ldots, x_s \in \mathfrak{m}$ is a d-sequence if for all $0 \leq i < j \leq s$,

$$(x_1, \ldots, x_i)M :_M x_j = (x_1, \ldots, x_i)M :_M x_{i+1}x_j,$$

$\underline{x}$ is called a strong d-sequence if any power $x_1^{n_1}, \ldots, x_s^{n_s}$ is a d-sequence and $\underline{x}$ is called an unconditioned strong d-sequence if it is a strong d-sequence for any order.

The aim of this short note is to give a new concept of sequences called dd-sequences, which is a slight generalization of the notion of unconditioned strong d-sequence. Then we apply these dd-sequences to study the polynomial property of the lengths of Koszul homology and of local cohomology modules with respect to the powers of a system of parameters. A completion of all proofs of statements given here can be found in [3].

2. dd-Sequences

Throughout this short note, let $(R, \mathfrak{m})$ denote a commutative local Noetherian ring and M a finitely generated R-module with $\dim(M) = d$. For a sequence $x_1, \ldots, x_s$ of elements of $\mathfrak{m}$ and an s-tuple of positive integers $n_1, \ldots, n_s$, we will denote by $\underline{x}$ the sequence $x_1, \ldots, x_s$ and $\underline{x}(\underline{n})$ the sequence $x_1^{n_1}, \ldots, x_s^{n_s}$.

Definition 2.1. *Let M be a finitely generated R-module and $\underline{x} = x_1, \ldots, x_s$ a sequence of elements of the maximal ideal $\mathfrak{m}$. We call $\underline{x}$ a dd-sequence of M if either $s = 1$, $\underline{x}$ is a d-sequence or $s > 1$ and*
(i) *$\underline{x}$ is a strong d-sequence,*
(ii) *for all $n > 0$ the sequence $x_1, \ldots, x_{s-1}$ is a dd-sequence of $M/x_d^n M$.*

It is easy to see that the sequence $\underline{x} = x_1, \ldots, x_s$ is a dd-sequence of M if and only if for all $i \in \{1, \ldots, s\}$ and all s-tuples of positive integers $n_1, \ldots, n_s$, the sequence $x_1^{n_1}, \ldots, x_i^{n_i}$ is a d-sequence of the module $M/(x_{i+1}^{n_{i+1}}, \ldots, x_s^{n_s})M$.

Let $x_1, \ldots, x_s$ be a sequence of elements of $\mathfrak{m}$. For all $i \in \{1, \ldots, s\}$, denote the sequence $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_s$ by $x_1, \ldots, \widehat{x_i}, \ldots, x_s$, the following proposition shows that dd-sequences are very relatively close to unconditioned strong d-sequences.

Proposition 2.2. *Let $x_1, \ldots, x_s$ be a dd-sequence of M, then for all $i \in \{1, \ldots, s\}$, the sequence $x_1, \ldots, \widehat{x_i}, \ldots, x_s$ is a dd-sequence of $M/x_i M$.*

As the first main result of this note, we have the following characterization of dd-sequence.

Theorem 2.3. *Let M be a finitely generated R-module, $\underline{x}$ a system of parameters of M. The system of parameters $\underline{x}$ is a dd-sequence if and only if*

$$l(M/\underline{x}(\underline{n})M) = \sum_{i=0}^{d} n_1 \ldots n_i e_i,$$

where $e_i = e(x_1, \ldots, x_i; (0 : x_{i+1})_{M/(x_{i+2} \ldots x_d)M})$ for $i < d$ and $e_d = e(\underline{x}, M)$.

In [6], the first author introduced the notion of a p-standard system of parameters and then this notion is used to solve the problem of Macaulayfication by T. Kawasaki in [10, 11]. Note that if $\underline{x}$ is a p-standard system of parameters of M then the function $l(M/\underline{x}(\underline{n})M)$ also has the form as mentioned in Theorem 2.3 (see [6, Theorem 2.6]). Therefore an immediate consequence of Theorem 2.3 is that every p-standard system of parameters is a dd-sequence.

Proposition 2.4. *Let $\underline{x}$ be a system of parameters of a finitely generated R-module M which is also a dd-sequence. Then $x_1^{n_1}, \ldots, x_d^{n_d}$ is a p-standard system of parameters for all $n_i \geq i$, $i = 1, \ldots, d$.*

Proposition 2.4 showed that the existence of these two kinds of systems of parameters are equivalent. However, a p-standard system of parameters is in general not necessary a dd-sequence as in the following example.

Example 2.5. Let R be the ring $k[[X_1, \ldots, X_{d+1}]]$ of formal power series over a field k with the maximal ideal $\mathfrak{m}$ and M denote the R-module R/IR where $I = (X_{d+1}^{d+1}, X_1X_{d+1}^d, X_2X_{d+1}^{d-1}, \ldots, X_dX_{d+1})$, $(d > 1)$. Then $\dim(M) = d$ and $X_1, \ldots, X_d$ is a system of parameters of M. By simple computation we get

$$\ell(M/(X_1^{n_1}, \ldots, X_d^{n_d})M) = \sum_{i=0}^{d} n_1 \ldots n_i,$$

where if $i = 0$ then $n_1 \ldots n_i = 1$. Hence the system of parameters $X_1, \ldots, X_d$ is a dd-sequence of M. On the other hand, it is not difficult to check that $H_{\mathfrak{m}}^0(M) = (X_{d+1}^d, X_2X_{d+1}^{d-1}, \ldots, X_dX_{d+1})/I$, thus $\mathfrak{a}_0(M) = \mathrm{Ann}_R(H_{\mathfrak{m}}^0(M)) = \mathfrak{m}$. Moreover, since R is an image of a regular local ring, we have by [4, Theorem 1.2] that $d-1 = \dim(R/\mathfrak{a}(M))$, where $\mathfrak{a}(M) = \mathfrak{a}_0(M) \ldots \mathfrak{a}_{d-1}(M)$ and $\mathfrak{a}_i(M) = \mathrm{Ann}_R(H_{\mathfrak{m}}^i(M))$. This leads to the existence of $i \in \{1, \ldots, d-1\}$ such that $\mathfrak{a}_i(M) \subseteq \mathfrak{m}$. Therefore $\mathfrak{a}(M) \subset \mathfrak{m}^2$. Since $X_d \notin \mathfrak{m}^2$ then $X_d \notin \mathfrak{a}(M)$. This shows that $X_1, \ldots, X_d$ is not a p-standard system of parameters of M.

3. Partial Euler-Poincaré Characteristics

Let $\underline{x}$ be a system of parameters of M. Denote by $K(\underline{x}; M)$ the Koszul complex of M with respect to $\underline{x}$ and $H_i(\underline{x}; M)$ its i-th homology module. The k-th Euler-Poincaré characteristic of $K(\underline{x}; M)$ is defined by

$$\chi_k(\underline{x}; M) = \sum_{i=k}^{d} (-1)^{i-k} l(H_i(\underline{x}; M)).$$

It is well-known that $\chi_0(\underline{x}; M) = e(\underline{x}; M)$, the multiplicity of M with respect to $\underline{x}$. Therefore, $\chi_1(\underline{x}; M) = l(M/\underline{x}M) - e(\underline{x}; M)$. In general, there are many examples showing that the function $\chi_k(\underline{x}(\underline{n}); M)$ is not a polynomial. However, if $\chi_k(\underline{x}(\underline{n}); M)$ is a polynomial then it is linear in each variable n_i, i.e

$$\chi_k(\underline{x}(\underline{n}); M) = \sum_{t=0}^{d} \sum_{0<i_1<\ldots<i_t\leq d} \lambda_{i_1\ldots i_t} n_{i_1} \ldots n_{i_t},$$

where the coefficients $\lambda_{i_1\ldots i_t}$ are integers.

It is known in [7] that for all $k \geq 0$ the function $\chi_k(\underline{x}(\underline{n}); M)$ is always bounded above by a polynomial and the least degree of these polynomials is independent of the choice of system of parameters, it is an invariant of M and denoted by $p_k(M)$. The invariant $p_1(M)$ was denoted in [5] by $p(M)$ and called the polynomial type of M. There are many interesting results on these invariants (see [5 - 7]). For examples, if we denote by $-\infty$ the degree of the polynomial zero then a module M is Cohen-Macaulay if and only if $p_1(M) = -\infty$ and M is generalized Cohen-Macaulay if and only if $p_1(M) \leq 0$. When the system of parameters $\underline{x}$ is a dd-sequence and $p_1(M) > 0$ then $p_1(M/x_1M) = p_1(M) - 1$. Therefore, if $p_1(M) = k$ then the module $M/(x_1, \ldots, x_k)M$ is generalized Cohen-Macaulay.

The following theorem is a criterion to check whether the function $\chi_k(\underline{x}(\underline{n}); M)$ is a polynomial.

Theorem 3.1. *Let M be a finitely generated R-module and $\underline{x}$ a system of parameters of M. Let n_0 be a positive integer. Then the function $\chi_k(\underline{x}(\underline{n}); M)$ is a polynomial for all $n_i \geq n_0$ if and only if for all $i = 1, 2, \ldots, d$ and $n_1, \ldots, n_d \geq n_0$, the following condition is satisfied*

$$(0 : x_i^{n_i})_{H_{k-1}(x_1^{n_1}, \ldots, \widehat{x_i^{n_i}}, \ldots, x_d^{n_d}; M)} = (0 : x_i^{n_0})_{H_{k-1}(x_1^{n_1}, \ldots, \widehat{x_i^{n_i}}, \ldots, x_d^{n_d}; M)}.$$

The main result of this note is the following theorem.

Theorem 3.2. *Let M be a finitely generated R-module, $\underline{x} = x_1, \ldots, x_d$ a system of parameters of M. Suppose that $\underline{x}$ is a dd-sequence of M. Then for all $k > 0$, the function $\chi_k(\underline{x}(\underline{n}); M)$ is a polynomial for all $n_1, \ldots, n_d > 0$. Moreover, the polynomial has the form as follows*

$$\chi_k(\underline{x}(\underline{n}); M) = \sum_{i=0}^{p_k(M)} n_1 \ldots n_i e(x_1, \ldots, x_i; (0 : x_{i+1})_{H_{k-1}(x_{i+2}, \ldots, x_d; M)}).$$

In order to prove the theorem, we need the following key lemma.

Lemma 3.3. *Let M be a finitely generated R-module and $\underline{x} = x_1, \ldots, x_s$ a sequence of elements of $\mathfrak{m}$. If $\underline{x}$ is a dd-sequence of M then for all $1 \leq i \leq j \leq s$ and $0 \leq k \leq s$ we have*

$$(0 : x_j)_{H_k(x_1, \ldots, x_{i-1}, x_{j+1}, \ldots, x_s; M)} = (0 : x_i x_j)_{H_k(x_1, \ldots, x_{i-1}, x_{j+1}, \ldots, x_s; M)}.$$

4. Local Cohomology Modules

Another consequence of Theorem 2.3 is that if $p_1(M) > 0$ and a system of parameters $\underline{x}$ of M is a dd-sequence then $p_1(M/x_1M) = p_1(M) - 1$. It follows that for $k \geq p_1(M)$, we have $p_1(M/(x_1^{n_1}, \ldots, x_k^{n_k})M) \leq 0$. Therefore the module $M_k = M/(x_1^{n_1}, \ldots, x_k^{n_k})M$ is a generalized Cohen-Macaulay module, hence $l(H^i_{\mathfrak{m}}(M_k)) < \infty$ for all $i < d - k$ and $x_{k+1}, \ldots, x_d$ is a standard system of parameters of M_k. In this section we are interested in the question: Whether $l(H^i_{\mathfrak{m}}(M_k))$ is a polynomial in $n_1, \ldots, n_k$?

Proposition 4.1. *Let $\underline{x}$ be a system of parameters of a finitely generated R-module M. If $\underline{x}$ is a strong d-sequence then the length of the Koszul homology module $H_i(x_1^{n_1}, \ldots, x_j^{n_j}; M)$ is finite and given by*

$$l(H_i(x_1^{n_1}, \ldots, x_j^{n_j}; M)) = \sum_{s=0}^{j-i} \binom{j-s-1}{i-1} l(H^0_{\mathfrak{m}}(M/(x_1^{n_1}, \ldots, x_s^{n_s})M)).$$

For a given system of parameters $\underline{x}$ which is a dd-sequence, we put

$$f_s(n_1, \ldots, n_s) = l((x_1^{n_1}, \ldots, x_s^{n_s})M :_M x_{s+1}/(x_1^{n_1}, \ldots, x_s^{n_s})M).$$

Therefore, by Proposition 4.1, we can show that

$$f_j(n_1, \ldots, n_j) = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i-1}{d-j-1} l(H_{d-i}(\underline{x}(\underline{n}); M)).$$

So by Theorem 3.2 the function $f_j(n_1, \ldots, n_j)$ is a polynomial for all j. Furthermore, if $k \geq p_1(M)$ then $x_{k+1}, \ldots, x_d$ is a standard system of parameters of $M_k = M/(x_1^{n_1}, \ldots, x_k^{n_k})M$. It follows that

$$\begin{aligned} l(H^i_{\mathfrak{m}}(M_k)) &=; \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} \ l(H^0_{\mathfrak{m}}(M_k/(x_{k+1}, \ldots, x_{k+j})M_k) \\ &= \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} \ f_{k+j}(n_1, \ldots, n_k, 1, \ldots, 1). \end{aligned}$$

We therefore get the following theorem

Theorem 4.2. *Let $\underline{x}$ be a system of parameters of a finitely generated R-module M. If $\underline{x}$ is a dd-sequence then the length $l(H^i_{\mathfrak{m}}(M/(x_1^{n_1}, \ldots, x_k^{n_k})M))$, considered as a function in variables $n_1, \ldots, n_k$, is a polynomial for all $k \geq p_1(M)$ and $i < d - k$.*

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