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The Property (LB_∞) and Fréchet-Valued Holomorphic Functions on Compact Sets

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Abstract. In this paper we establish an equivalence between weakly holomorphic and holomorphic extension for Fréchet-valued functions defined on compact subsets in a Fréchet space concerning with the linear topological invariant (LB_∞) introduced by Vogt [14]. Next, using the linear topological invariant (LB_∞) we extend a recent result of Arendt - Nikolski [1] for Fréchet-valued functions on open subsets in Schwartz - Fréchet spaces.

1. Introduction

Vector - valued holomorphic functions are very useful, for example, in the theory of one-parameter semigroups or in spectral theory. However, when proving theorems about scalar-valued holomorphic functions, it is sometimes useful to consider functions with values in a Banach space (see [1]). In order to verify the holomorphicity of valued - vector functions on open subsets in a Fréchet space one only need to check its weak holomorphicity. But the situation is quite different in the case of compact subsets. Our first aim in the paper is to look for certain conditions for Fréchet spaces E and F such that every F-valued weakly holomorphic germs on compact sets of E is holomorphic. More precisely if $H(K,F)$ (resp. $H_w(K,F)$) denotes the space of F-valued holomorphic (resp. weakly holomorphic) germs on a compact subset $K \subset E$ then we look for conditions for E and F such that the following holds

$$H_w(K,F) = H(K,F). \tag{1}$$

This problem was considered by some authors. For example, by using the closed graph theorem, Waelbroeck in [16] has shown that (1) holds in the case where

E, F are Banach spaces and K is a set of uniqueness. When K is a regular compact set in $\mathbb{C}^n$, the equality (1) was proved by Siciak [13]. Recently, Hai and Khue [7] using again the closed graph theorem, proved that, for a fixed Fréchet space F, the equality (1) holds for every LB^∞ - regular compact subset K in a Fréchet space E if and only if F has property (DN).

In the present paper first we prove the following

Theorem 1.1. *Let F be a Fréchet space. Then F has property (LB_∞) if and only if*

$$H_w(K,F) = H(K,F)$$

holds for every compact set of uniqueness in every Fréchet space E having property (Ω).

Similarly, we give conditions under which the above equality holds for every compact subset K in an arbitrary Fréchet space E. Namely we have

Theorem 1.2. *Let F be a Fréchet space. If F has property (LB_∞) then*

$$H_w(K,F) = H(K,F)$$

holds for every convex compact set K of the strong uniqueness in every Fréchet space E.

Note that when defining the "weak notion" for a Fréchet - valued function we require that for all $u \in F'$, the function uf should have the same property. However, recently Arendt and Nikolski in [1], have given the notion of $\sigma(F,W)$-holomorphic functions for the case F is a Banach space, where W is a subspace of F' defining the topology of F. They have established a relation between the $\sigma(F,W)$ - holomorphicity and the holomorphicity of a function $f : D \to F$ in the case $D \subset \mathbb{C}$ is an open subset and F is a Banach space (see Theorem 1.8 in [1]). The fourth section of this paper will deal with an extension of the above mentioned result of Arendt - Nikolski for the Fréchet case when D is an open set either in a Schwartz - Fréchet space or in $\mathbb{C}$.

We have the following theorem

Theorem 1.3. *Let E and F be Schwartz - Fréchet spaces, D an open subset in E and $f : D \to F$. Assume that E has property (Ω), F has property (LB_∞) and f is a $\sigma(F,W)$ - holomorphic function for a subspace W of F' defining the topology of F. Then f is holomorphic on a dense open subset of D.*

In case D is an open subset in $\mathbb{C}$, the hypothesis of the Schwartz property of F in Theorem 1.3 is superfluous. Namely we have

Theorem 1.4. *Let $f : D \to F$ be a $\sigma(F,W)$ - holomorphic function where D is an open set in $\mathbb{C}, W \subset F'$ is a subspace defining the topology of F and F a Fréchet space having property (LB_∞). Then f is holomorphic on a dense open subset of D.*

In the final section of this paper we give some examples concerning with Theorems 1.3 and 1.4. The first example shows that Theorem 1.3 is not true for Banach-valued analytic functions on $\mathbb{R}$. While the second one says that the converse of Theorem 1.4 can not happen.

Our paper is organized as follows. Beside the introduction, the second section is devoted to give some notions and to recall the linear topological invariants (LB_∞) and (Ω). The third section contains the proof of Theorem 1.1 and 1.2 while the Theorems 1.3 and 1.4 are proved in the fourth section. The examples concerning Theorems 1.3 and 1.4 are given in the fifth section.

2. Preliminaries

All Fréchet spaces considered in the paper are assumed to be complex.

2.1. Let E, F be Fréchet spaces and K a compact set in E. A function $f : K \to F$ is called holomorphic if it can be holomorphically extended to a neighborhood of K in E.

If for all $u \in F'$, the topological dual space of F, the function $u.f : K \to \mathbb{C}$ is holomorphic then we say that f is weakly holomorphic. We write $H(K, F)$ (resp. $H_w(K.F)$) for the space of holomorphic (resp. weakly holomorphic) functions on K with values in F. In case $F = \mathbb{C}$, we write $H(K)$ instead of $H(K, \mathbb{C})$. The space $H(K)$ is equipped with the inductive limit topology. Namely

$$H(K) = \lim_{\substack{U \supset K \\ U \text{ open}}} \operatorname{ind} H^\infty(U),$$

where $H^\infty(U)$ is the Banach space of bounded holomorphic functions on U. It is known [4] that $H(K)$ is regular, i.e. for every bounded set B in $H(K)$ there exists a neighborhood U of K such that B is contained and bounded in $H^\infty(U)$. Moreover if E is a Schwartz - Fréchet space then so is $[H(K)]'_\beta$, the dual space of $H(K)$ equipped with the strong topology. In particular, $[H(K)]'_\beta$ is reflexive.

2.2. Let E, F be Fréchet spaces and $D \subset E$ an open set, $W \subset F'$ and $f : D \to F$ be a F-valued function on D. f is called $\sigma(F, W)$ - holomorphic if $u.f$ is holomorphic for all $u \in W$.

2.3. A subset W of F' is called defining topology if the topology of F is the uniformly convergent topology on bounded subsets of F' contained in W.

2.4. Now we deal with linear topological invariants introduced and investigated by Vogt (see [14, 15]) on Fréchet spaces.

Let F be a Fréchet space with the topology defined by an increasing sequence of semi-norms $\{\|.\|_k\}$. For each $k \geq 1$, and $u \in F'$, let

$$\|u\|_k^* = \sup\{|u(x)| : x \in U_k\},$$

where

$$U_k = \{x \in F : \|x\|_k \leq 1\}.$$

We consider the following properties for F

(LB_∞) $\forall\{\rho_n\}\uparrow+\infty,\ \exists p\geq 1,\ \forall q\geq 1,\ \exists k_q\geq q,\ \ C>0,\ \forall x\in F,\ \exists q\leq m\leq k_q$:

$$\|x\|_q^{1+\rho_m}\leq C\|x\|_m\|x\|_p^{\rho_m};$$

(Ω) $\forall p\geq 1,\ \exists q\geq 1,\ \forall k\geq 1,\ \exists d>0,\ C>0,\ \forall u\in F'$

$$\|u\|_q^{*1+d}\leq C\|u\|_k^*\|u\|_p^{*d}.$$

Throughout this paper we write $F\in(\Omega)$ (resp. $F\in(LB_\infty)$) whenever F has property (Ω) (resp. (LB_∞)).

2.5. Let K be a compact set in a Fréchet space E. K is called a set of uniqueness if for every $f\in H(K)$, $f\big|_K=0$ then $f=0$ on some neighborhood of K in E.

Next, K is called a set of the strong uniqueness if for each absolutely convex compact subset $B\subset E$ containing K there exists an absolutely convex compact $\widetilde{B}\subset E$, $B\subset\widetilde{B}$ such that K is a compact set of uniqueness in $E(\widetilde{B})$, the Banach space induced by $\widetilde{B}$.

3. Frechet-Valued Holomorphic Germs on Compact Sets

In this section we give some results about the equivalence between weak holomorphicity and holomorphicity of Fréchet-valued functions on compact sets in a Fréchet space.

First we prove the following

Theorem 3.1. *Let F be a Fréchet space. Then F has property (LB_∞) if and only if*

$$H_w(K,F)=H(K,F)$$

holds for every compact subset K of uniqueness in every Fréchet space E having property (Ω).

In order to prove Theorem 3.1 we state the following result which was proved in [3].

Proposition 3.2. *Let F be a Fréchet space having property (LB_∞). Then $(F'_{bor})'_\beta$ and, in particular, F''_β also has property (LB_∞), where F'_{bor} denotes the dual space F' equipped with the bornological topology associated to the strong topology β.*

Proof of Theorem 3.1.
Necessity. Let E, F and K be as in the statement of the theorem. It suffices to show that

$$H_w(K,F)\subset H(K,F).$$

Given $f\in H_w(K,F)$. By the uniqueness of K we can define the linear map $S:F'_{bor}\to H(K)$ by

$$S(u) = \widehat{uf}, \quad \text{for} \quad u \in F'_{bor},$$

where F'_{bor} denotes the space F' equipped with the bornological topology associated to the strong topology $\beta = \beta(F', F)$ on F' and $\widehat{uf}$ is the unique holomorphic extension of uf to a neighborhood of K in E. Again by the uniqueness of K we deduce that S has the closed graph. By [6], S is continuous.

On the other hand, since E has property (Ω) then so does $[H(K)]'_\beta$ [9]. From [15] there exists an index set I such that $[H(K)]'_\beta$ is isomorphic to a quotient space of

$$\Lambda_\infty(\alpha, \ell^1(I)) = \big\{(x_n) \subset \ell^1(I) : \sum_{n=1}^{\infty} \|x_n\| n^k < +\infty \ \forall k \geq 1\big\},$$

where $\alpha = \big(\log(n+1)\big)_{n\geq 1}$. However, if $F \in (LB_\infty)$ then by Proposition 3.2, so does $\big[F'_{bor}\big]'_\beta$. From $\sup\limits_{n\geq 1} \dfrac{\log(n+1)}{\log n} < \infty$ and repeating the proof of Theorem 3.2 of Vogt [14] we deduce that every continuous linear map from $\Lambda_\infty(\alpha, \ell^1(I))$ and, hence, from $[H(K)]'_\beta$ to $\big[F'_{bor}\big]'_\beta$, is bounded on a neighborhood of $0 \in [H(K)]'_\beta$. Now the same argument as in the proof of Theorem 2.1 in [7] shows that there exist a neighborhood V of K in E and a holomorphic function $g : V \to F$ with $g\big|_K = f$. Hence $f \in H(K, F)$.

Sufficiency. Take $E = \mathbb{C}$; then by [12, p. 143] there exists a compact polar set of uniqueness $K \subset \mathbb{C}$. Then by a result of Zaharjuta [17], $H(K) = H(\{0\})$. Hence

$$[H(K)'_\beta \cong [H(\{0\})]'_\beta \cong H(\mathbb{C}_\infty \setminus \{0\}) \cong H(\mathbb{C}).$$

The second isomorphism is a result of the so-called Grothendieck – Köthe – Silva duality. From Theorem 3.2 of Vogt [15] we deduce that F has property (LB_∞) if we show that

$$L\big([H(K)]'_\beta, F\big) = LB\big([H(K)]'_\beta, F\big).$$

Let $S \in L\big([H(K)]'_\beta, F\big)$ be given. Then we define the function

$$f : K \longrightarrow F$$

by $f(z) = S(\delta_z)$, where $\delta_z(h) = h(z)$ for $z \in K$ and $h \in H(K)$. It is easy to see that $f \in H_w(K, F)$. By the hypothesis we can find a neighborhood U of K in $\mathbb{C}$ and *a* F-valued holomorphic function $\widehat{f} : U \to F$ such that $\hat{f}\big|_K = f$. By shrinking U we may assume that $B = \widehat{f}(U)$ is bounded in F. Then

$$|S'(u)(z)| = |\widehat{f}(z)(u)| \leq 1$$

for $z \in U$, $u \in B^0$, where B^0 denotes the polar of B in F'_β and S' is the adjugate map of S. Hence, $S'(B^0)$ is contained and bounded in $H^\infty(U)$. Thus it is bounded in $H(K)$. Put $C = \big(S'(B^0)\big)^0$. Then C is a neighborhood of $0 \in [H(K)]'_\beta$. Now for $u \in B^0$, $t \in C$ we have

$$|S(t)(u)| = |S'(u)(t)| \leq 1.$$

Thus $S(C) \subset B^{00} \cap F$ and, thereby, $S(C)$ is bounded in F, which completes the proof of Theorem 3.1. ∎

Now we establish a sufficient condition for the equality (1) without the assumption of property (Ω) on E. We have the following.

Theorem 3.3. *Let F be a Fréchet space. If F has property (LB_∞), then*

$$H_w(K,F) = H(K,F)$$

holds for every convex compact set K of the strong uniqueness in every Fréchet space E.

Proof. Given K a convex compact set of the strong uniqueness. It suffices to show that

$$H_w(K,F) \subset H(K,F).$$

Let $f \in H_w(K,F)$. From the hypothesis, without loss of generality, we can assume that if $B \subset E$ is an absolutely convex compact subset containing K, then K is a compact set of uniqueness in $E(B)$, the Banach space induced by B. Put

$$\mathcal{B}_K(E) = \{B : B \text{ is an absolutely convex compact subset of } E \text{ contaning } K\}.$$

For each $B \in \mathcal{B}_K(E)$ we denote

$$H_B(K) = \lim\,\mathrm{ind}\{H^\infty(W) : W \text{ is a neighborhood of } K \text{ in } E(B)\}.$$

We may assume that W is absolutely convex. From the strong uniqueness of K we can define the linear map

$$S_B : F'_{bor} \longrightarrow H_B(K)$$

by

$$S_B(u) = \widehat{uf},$$

where $\widehat{uf}$ is a holomorphic extension of uf to a neighborhood of K in $E(B)$. Again using the strong uniqueness of K in $E(B)$ we deduce that S_B has the closed graph and, hence, it is continuous. As in the proof of Theorem 3.1, in view of $[H_B(K)]'_\beta \in (\Omega)$ and $[F'_{bor}]'_\beta \in (LB_\infty)$ it implies that there exist an absolutely convex neighborhood W_B of K in $E(B)$ and a holomorphic function $\hat{f}_B : W_B \to F$ such that $\hat{f}_B\big|_K = f$.If necessary we may increase W_B and, hence, we can assume that $B \subset W_B$. Put

$$W = \bigcup_{B \in \mathcal{B}_K(E)} W_B$$

and define the function $\hat{f} : W \to F$ given by

$$\hat{f}\big|_{W_B} = \hat{f}_B.$$

Now we can check that $\hat{f}$ defined above is correct and is holomorphic on $\operatorname{Int} W$ which is taken in E. Let B_1, B_2 be in $\mathcal{B}_K(E)$. Without loss of generality we may assume that $B_1 \subset B_2$. Thus $E(B_1) \hookrightarrow E(B_2)$ and $W_{B_2} \cap W_{B_1}$ is open and convex in $E(B_1)$. Moreover, $\hat{f}_{B_1}$ and $\hat{f}_{B_2}$ are holomorphic on $W_{B_2} \cap W_{B_1} \supset K$. Notice that $\hat{f}_{B_1}\big|_K = f = \hat{f}_{B_2}\big|_K$ and, once more, from the uniqueness of K we deduce that

$$\hat{f}_{B_1}\big|_{W_{B_2} \cap W_{B_1}} = \hat{f}_{B_2}\big|_{W_{B_2} \cap W_{B_1}}.$$

Observe that $\hat{f}$ is holomorphic on $\operatorname{Int} W$, since E is a Fréchet space and $\hat{f}\big|_{W_B}$ is holomorphic for all $B \in \mathcal{B}_K(E)$. It remains to check that $K \subset \operatorname{Int} W$. Assume that there exist $z_0 \in K$ and a sequence $\{z_n\} \subset E$, $z_n \to z_0$ but $z_n \notin W$ for $n \geq 1$. Consider $B = \overline{\operatorname{conv}}\big\{K, \{z_n\}, z_0\big\}$, where $\overline{\operatorname{conv}}$ denotes the closure of the absolutely convex hull of $\big\{K, \{z_n\}, z_0\big\}$. By [5] we can find an absolutely convex compact subset $\widetilde{B} \subset E$, $B \subset \widetilde{B}$ and B is compact in $E(\widetilde{B})$. Then $\widetilde{B} \in \mathcal{B}_K(E)$ and since B is compact in $E(\widetilde{B})$, without loss of generality, we may assume that $\{z_n\} \to z_0$ in $E(\widetilde{B})$. Thus there exists n_0 such that for $n > n_0$ $\{z_n\} \subset W_{\widetilde{B}} \subset W$. This is absurd. ∎

4. $\sigma(F, W)$ - Holomorphic Functions

The aim of the section is to give an extension of a result of Arendt – Nikolski [1] for the Fréchet case in the relation with property (LB_∞).

First we state the following

Theorem 4.1. *Let E, F be Schwartz – Fréchet spaces, D an open subset in E and $f : D \to F$. Assume that E has property (Ω), F has property (LB_∞) and f is $\sigma(F, W)$-holomorphic for a subspace W of F'_β defining the topology of F. Then f is holomorphic on a dense open subset of D.*

Note that in the case F is a Banach space and $D \subset \mathbb{C}$ the above theorem was proved by Arendt and Nikolski [1]. In fact this result of Arendt - Nikolski still holds true if D is an open subset in an arbitrary Fréchet space.

Proof. Given $f : D \to F$ as in the statement of the theorem. For each $k \geq 1$ consider the canonical map $\omega_k : F \to F_k$, the Banach space associated to the semi-norm $\|.\|_k$ on F. Since $W \subset F'$ defines the topology of F, then $W_k = W \cap F'_k$ defines the topology of F_k. Using the above mentioned result of Arendt – Nikolski there exists a dense open subset $D_k \subset D$ such that $w_k f$ is holomorphic on D_k. The Baireness of D implies that $\widetilde{D} = \bigcap_{k \geq 1} D_k$ is also dense in D.

Thus for each $z \in \widetilde{D}$ we can define the linear map

$$S^z : F' \longrightarrow H(z)$$

given by

$$S^z(u) = (uf)_z,$$

where $(uf)_z$ denotes the germ of uf at z. We show that S^z has the closed graph. Indeed, since $F' \times H(z)$ is the strongly dual space of the Fréchet space $F \times [H(z)]'$, it suffices to check that the intersection of the graph Γ_z of S^z with every equicontinuous subset of $F' \times H(z)$ is weakly closed in it. Given Z such a subset of $F' \times H(z)$. Choose $k \geq 1$ and a neighborhood V of z in D_k such that $Z \subset U_k^0 \times \{\varphi \in H^\infty(V) : \|\varphi\| \leq c\}$. Let $\{(u_\alpha, S^z u_\alpha)\} \subset \Gamma_z \cap Z$ which is weakly convergent to $(u, \varphi) \in Z$. Since $H(z)$ is a DFS-space we can find a neighborhood V_1 of z in V such that the restriction map $H^\infty(V) \to H^\infty(V_1)$ is compact. This implies that $\{S^z u_\alpha\}$ is convergent to g in $H^\infty(V_1)$ and, consequently,

$$uf\big|_{V_1} = g\big|_{V_1} = \varphi\big|_{V_1}.$$

Hence, $(u, \varphi) \in \Gamma_z$. By the closed graph theorem S^z is continuous. Now using the hypothesis $E \in (\Omega)$ and, hence $[H(z)]' \in (\Omega)$ [9], $F \in (LB_\infty)$ and by the same argument as in the proof of Theorem 3.1 (the necessary condition) we infer that f is holomorphic on a neighborhood of z in D. Thereby the theorem is proved.

■

Now we will give a result similar to the above theorem when D is open in $\mathbb{C}$. In this case, the hypothesis of the Schwartz property for F is superfluous.

We have the following

Theorem 4.2. *Let $f : D \to F$ be a $\sigma(F, W)$-holomorphic function where D is an open set in $\mathbb{C}$ and F a Fréchet space having property (LB_∞). Then f is holomorphic on a dense open subset of D.*

Proof. In notations of Theorem 4.1 we prove that f is holomorphic at every $z \in \widetilde{D}$. We may assume that z is a limit point of $\widetilde{D}$. Choose a sequence $\{z_n\} \subset \widetilde{D}$ converging to z. Consider the compact set $K = \{z_n, z\}$. Let $\{V_n\}_{n \geq 1}$ be a decreasing neighborhood basis of z such that $z_k \notin \partial V_n$ for all k, $n \geq 1$. Then $Y_n = K \setminus V_n$ is finite and we have an exact sequence

$$0 \to \bigoplus_{n=1}^{\infty} H^\infty(Y_n)\big/A(Y_n) \xrightarrow{\oplus e_n} H(K)\big/A(K) \xrightarrow{R} H(z) \to 0 \tag{3}$$

with

$$H^\infty(Y_n)\big/A(Y_n) \cong \mathbb{C}^{k_n},$$

$$k_n = \#Y_n \quad \text{for} \quad n \geq 1,$$

where

$$A(Y_n) = \{\varphi \in H^\infty(Y_n) : \varphi\big|_{Y_n} = 0\},$$
$$A(K) = \{\varphi \in H(K) : \varphi\big|_K = 0\}.$$

We explain the existence of this exact sequence. For each $\varphi \in H^\infty(Y_n)$ we define the element $e_n(\varphi) \in H(K)$ as follows. Since $Y_n = K \setminus V_n$ then we can find two disjoint neighborhoods U_1 and U_2 of $K \cap V_n$ and $K \setminus V_n$ respectively such that $\varphi \in H^\infty(U_2)$. Then we put

$$e_n(\varphi) = \begin{cases} \varphi & \text{on } U_2, \\ 0 & \text{on } U_1, \end{cases}$$

and define $R(\varphi + A(K)) = \varphi_z$. Obviously, from the definition of R and $\oplus e_n$ we claim that the above sequence (3) is exact. On the other hand, for each $u \in F'_\beta$ the function uf is holomorphic on $\widetilde{D}$. Then we may define the linear map

$$S : F'_{bor} \longrightarrow H(K)/A(K)$$

given by

$$S(u) = uf + A(K), \quad u \in F'_{bor}.$$

It is easy to see that S has a closed graph. Hence it is continuous. Since F has property (LB_∞) and $[H(z)]' \in (\Omega)$ [9], by Theorem 3.2 of Vogt [14] we can find a continuous semi-norm ρ on F' such that RS is factorized through the Banach space F'_ρ associated to ρ. Let $T : F'_\rho \to H(z)$ be a continuous linear map such that

$$R.S = T\omega_\rho,$$

where $\omega_\rho : F' \to F'_\rho$ is the canonical map. Since $R : H(K)/A(K) \to H(z)$ is surjective and $H(z)$ is nuclear, then there exists a continuous linear map $S_1 : F'_\rho \to H(K)/A(K)$ such that

$$T = R.S_1.$$

Consider the continuous linear map

$$S - S_1\omega_\rho : F' \to \ker R \cong \mathbb{C}^N.$$

Since F has a continuous norm, it is easy to see that $S - S_1\omega_\rho$ can be factorized through F'_{ρ_1}, where ρ_1 is a continuous semi-norm on F' with $\rho \le \rho_1$. We may assume that $\rho = \rho_1$. Let $G : F'_\rho \to \ker R$ be a continuous linear map with

$$S - S_1\omega_\rho = G\omega_\rho,$$

or

$$S = (S_1 + G)\omega_\rho.$$

Hence S is factorized through F'_ρ. On the other hand, since $\omega_K : H(K) \longrightarrow H(K)/A(K)$ is surjective and $H(K)/A(K)$ is a DFN-space, then there exists a continuous linear map $\hat{S} : F'_\rho \longrightarrow H(K)$ such that

$$S = \omega_k \hat{S} \omega_\rho.$$

Choose a neighborhood V of K in $\mathbb{C}$ such that $\hat{S}$ continuously maps F'_ρ into $H^\infty(V)$. Then the form

$$\widehat{F^z}(\lambda)(u) = \hat{S}\omega_\rho(u)(\lambda)$$

for $\lambda \in V$, $u \in F'$ defines a holomorphic extension of $f\big|_K$ to V. By the identity theorem $\big(\widehat{f^z}\big)_z$ is independent on sequences $\{z_n\}$ converging to z. By Δ_z we denote the disc centered at z such that $\widehat{f^z}$ is holomorphic on Δ_z and

$$\omega_p \widehat{f^z}\big|_{\Delta_z} = \omega_p f\big|_{\Delta_z},$$

where $\| \ \|_p$ is a continuous norm on F in the definition of property (LB_∞). Hence

$$\widehat{f^z}\big|_{\Delta_z \cap \widetilde{D}} = f\big|_{\Delta_z \cap \widetilde{D}}.$$

This yields that

$$\widehat{f^{z_1}}\big|_{\Delta_{z_1} \cap \Delta_{z_2} \cap \widetilde{D}} = f\big|_{\Delta_{z_1} \cap \Delta_{z_2} \cap \widetilde{D}} = \widehat{f^{z_2}}\big|_{\Delta_{z_1} \cap \Delta_{z_2} \cap \widetilde{D}}.$$

Since $\widetilde{D}$ is dense in D and, hence, $\Delta_{z_1} \cap \Delta_{z_2} \cap \widetilde{D}$ is dense in $\Delta_{z_1} \cap \Delta_{z_2}$, then

$$\widehat{f^{z_1}}\big|_{\Delta_{z_1} \cap \Delta_{z_2}} = \widehat{f^{z_2}}\big|_{\Delta_{z_1} \cap \Delta_{z_2}} \quad \text{for all } z_1, z_2 \in \widetilde{D}.$$

Thus the family $\{\widehat{f^z}\}_{z \in \widetilde{D}}$ defines a holomorphic function $\hat{f} : U \to F$ for which $\hat{f}\big|_{\widetilde{D}} = f\big|_{\widetilde{D}}$, where $U = \bigcup_{z \in \widetilde{D}} \Delta_z$ is a neighborhood of $\widetilde{D}$ in D. Put $\widetilde{U} = U \cap D_p$. Then $\widetilde{U}$ is open and dense in D and

$$\omega_p \hat{f}(x) = \omega_p f(x)$$

for $x \in \widetilde{U}$. Hence $f = \hat{f}\big|_{\widetilde{U}}$ is holomorphic on $\widetilde{U}$. ■

Remark. We do not know if the above theorem is true for $D \subset C^n$ with $n \geq 2$.

5. Some Examples

In this section we give some examples concerning Theorems 4.1 and 4.2.

5.1. First we recall the notion of Fréchet-valued analytic functions. Let $\Omega \subset \mathbb{R}^n$ be an open subset and $f : \Omega \to F$ be a function with values in a Fréchet space F. f is called a real function on Ω if locally f is a restriction of a F-valued holomorphic function. Since Theorem 4.1 is valid for Fréchet-valued holomorphic functions then, this theorem may be true for Fréchet-valued analytic functions. But, in fact, the following example shows that Theorem 4.1 is not true even for Banach-valued analytic functions. Namely we show that there exists a function $f : \mathbb{R} \to c_0$ such that $u.f$ is analytic for all $u \in W \subset c_0'$ defining the topology of c_0 , but f is not analytic at any point of $\mathbb{R}$.

Indeed, let $\{\xi_n\}$ be a dense sequence in $\mathbb{R}$. Consider the function

$$f : \mathbb{R} \longrightarrow c_0$$

given by

$$f(x) = \Big(\frac{1}{n(1 + n(x - \xi_n)^2)}\Big)_{n \geq 1}.$$

Let $W = \operatorname{span}\{e_n\} \subset c_0' = \ell_1$ where $e_n = \big(\underbrace{0, 0, \dots, 0, 1}_{n}, 0, \dots\big)$. Then W defines the topology of c_0. Moreover, for every $u \in W$, $u = \lambda_1 e_{n_1} + \cdots + \lambda_k e_{n_k}$ we have

$$uf(x) = \frac{\lambda_1}{n_1(1 + (x - \xi_{n_1}))^2} + \frac{\lambda_2}{n_2(1 + (x - \xi_{n_2})^2)} + \cdots + \frac{\lambda_k}{n_k(1 + (x - \xi_{n_k})^2)}.$$

Hence $uf(x)$ is real analytic on $\mathbb{R}$ for every $u \in W$. However, f is not real analytic at ξ_n for all $n \geq 1$ and, hence, f is not real analytic on every non-empty open subset of $\mathbb{R}$. Indeed, let f be real analytic at ξ_{n_0}. Choose $\delta > 0$ such that

$$f(x) = \sum_{k=0}^{\infty} a_k (x - \xi_{n_0})^k$$

for $|x - \xi_{n_0}| < \delta$, where $a_k \in c_0$ for $k \geq 1$. Hence

$$g(z) = \sum_{k=0}^{\infty} a_k (z - \xi_{n_0})^k, \quad |z - \xi_{n_0}| < \delta,$$

is a holomorphic extension of $f\big|_{\{|x - \xi_{n_0}| < \delta\}}$. There exists j_0 such that for $j > j_0$ then $|\xi_{n_j} - \xi_{n_0}| < \dfrac{\delta}{3}$. For $j > j_0$, the closed disc $\overline{\Delta}\big(\xi_{n_j}, \dfrac{\delta}{3}\big) \subset \Delta(\xi_{n_0}, \delta)$. Hence

$$g(z) = \sum_{k=0}^{\infty} a_k^j (z - \xi_{n_j})^k$$

for $|z - \xi_{n_j}| \leq \dfrac{\delta}{3}$, $a_k^j \in c_0$ for all $k \geq 0$. Replacing z by x with $|x - \xi_{n_j}| \leq \dfrac{\delta}{3}$ we have

$$f(x) = \sum_{k=0}^{\infty} a_k^j (x - \xi_{n_j})^k,$$

where the series is convergent when $|x - \xi_{n_j}| \leq \dfrac{\delta}{3}$. Write

$$a_k^j = \big(a_{k1}^j, a_{k2}^j, \dots, a_{kn_j}^j, \dots\big)$$

for all $k \geq 0$ and compare with the n_j-component of $f(x)$ we derive that

$$\frac{1}{n_j(1 + n_j(x - \xi_{n_j})^2)} = \sum_{k=0}^{\infty} a_{kn_j}^j (x - \xi_{n_j})^k, \quad |x - \xi_{n_j}| \leq \frac{\delta}{3}.$$

Hence

$$\sum_{m=0}^{\infty} (-1)^m n_j^{m-1} (x - \xi_{n_j})^{2m} < +\infty$$

for $|x - \xi_{n_j}| \leq \dfrac{\delta}{3}$ and $j > j_0$. Thus

$$\sum_{m=0}^{\infty} (-1)^m n_j^{m-1} \Big(\frac{\delta}{3}\Big)^{2m} < +\infty$$

for $j > j_0$ and this is impossible.

5.2. Now we give an example showing that the converse of Theorem 4.2 is not true.

Consider the Fréchet space ω of all sequences of complex numbers. A subspace W of $\omega' = \oplus\mathbb{C}$ is called separating if for all $x \in \omega \setminus \{0\}$ there exists $\varphi \in W$ such that $\varphi(x) \neq 0$. By the Hahn – Banach theorem it is easy to see that if $W \subset \omega'$ separates then W is $\sigma(\omega', \omega)$-dense in ω'. On the other hand, since every linear functional on $\omega' = \oplus\mathbb{C}$ is continuous it follows that if $W \subset \omega'$ is a subspace then W is $\sigma(\omega', \omega)$-closed in ω'. However, if $W \subset \omega'$ is a subspace defining the topology of ω then it is separating and by the above argument $W = \omega'$. Hence, if Ω is an open subset of a Fréchet space E and $f : \Omega \to \omega$ is $\sigma(\omega, W)$-holomorphic where $W \subset \omega'$ is a subspace defining the topology of ω then f is holomorphic. However ω does not have property (LB_∞).

5.3. Finally, a question posed here is that: Is Theorem 4.2 valid for any Fréchet space? We give an example related to this question.

Assume that B is an infinite-dimensional Banach space. Then there exists a subspace W of B' defining the topology of B but it does not determine boundedness [1]. Take a dense subset $\{z_n\}$ of the unit disc Δ. By [1] we can find a $\sigma(B, W)$ - holomorphic function $h_1 : \Delta \to B$ such that h_1 is not holomorphic at z_1. Putting $h_2 = h_1\theta_1$ where $\theta_1 : \Delta \to \Delta$ is a biholomorphism with $\theta_1(z_2) = z_1$ we deduce that there exists a $\sigma(B, W)$-holomorphic function $h_2 : \Delta \to B$ such that h_2 is not holomorphic at z_2. Continuing this process we obtain a sequence $\{h_n\}$ of $\sigma(B, W)$-holomorphic functions $h_n : \Delta \to B$ such that h_n is not holomorphic at z_n. Then $h = \{h_n\}$: $\Delta \to B^{\mathbb{N}}$ is a $\sigma(B^{\mathbb{N}}, \oplus W)$ - holomorphic function which is not holomorphic at every z_n. Hence h is not holomorphic on Δ.

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