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Quasi-Permutation Representations of $F_{p,q}$

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Abstract. In [1], we gave algorithms to calculate $c(G)$, $q(G)$ and $p(G)$ for a finite group G. In this paper, we will show that

$$c(F_{p,q}) = q(F_{p,q}) = p(F_{p,q}) = p,$$

where $F_{p,q} = \langle a, b : a^p = b^q = 1, b^{-1}ab = a^r \rangle$, p is a prime number, $q \mid p-1$ and q is an integer number.

1. Introduction

By a quasi-permutation matrix we mean a square matrix over the complex field $\mathbb{C}$ with non-negative integral trace. Thus every permutation matrix over $\mathbb{C}$ is a quasi-permutation matrix. For a given finite group G, let $p(G)$ denote the minimal degree of a faithful permutation representation of G (or of a faithful representation of G by permutation matrices), let $q(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices over the rational field $\mathbb{Q}$, and let $c(G)$ denote the minimal degree of a faithful representation of G by complex quasi-permutation matrices. (See [1]). It is easy to see that

$$c(G) \leq q(G) \leq p(G)$$

where G is a finite group.

In this paper, we will assume that $G = F_{p,q}$ where p is a prime, q is an integer number dividing $p-1$. Then we will calculate $p(G)$, $q(G)$ and $c(G)$. We will show that

$$c(G) = q(G) = p(G) = p.$$

2. Algorithms for $p(G)$, $c(G)$ and $q(G)$

Lemma 2.1. *Let G be a finite group with a unique minimal normal subgroup. Then $p(G)$ is the smallest index of a subgroup with trivial core (that is, containing no non-trivial normal subgroup).*

Proof. See [1, Corolary 2.4]. ∎

Theorem 2.2. *Let G be a finite group. Then*

$$p(G) = \min\{\sum_{i=1}^{n} |G : H_i| : H_i \leq G \quad for \quad i = 1, 2, \ldots, n \quad and \quad \bigcap_{i=1}^{n} \bigcap_{x \in G} H_i^x = 1\}.$$

Proof. See [1, Theorem 2.2]. ∎

Definition 2.3. *Let χ be a character of G such that, for all $g \in G$, $\chi(g) \in \mathbb{Q}$ and $\chi(g) \geq 0$. Then we say that χ is a non-negative rational valued character.*

Notation. Let $\Gamma(\chi)$ be the Galois group of $\mathbb{Q}(\chi)$ over $\mathbb{Q}$.

Definition 2.4. *Let G be a finite group and let χ be an irreducible complex character of G. Then define*

(1) $d(\chi) = |\Gamma(\chi)|\,\chi(1)$;

(2) $m(\chi) = \begin{cases} 0 & if \quad \chi = 1_G \\ \left|\min\{\sum_{\alpha \in \Gamma(\chi)} \chi^{\alpha}(g) : g \in G\}\right| & otherwise \end{cases}$;

(3) $c(\chi) = \sum_{\alpha \in \Gamma(\chi)} \chi^{\alpha} + m(\chi) 1_G$.

Lemma 2.5. *Let G be a finite group with a unique minimal normal subgroup. Let $m_{\mathbb{Q}}(\chi)$ denote the Schur index of χ in $\mathbb{Q}$. Then there exists a faithful irreducible complex character of G and*

(1) $c(G) = \min\{c(\chi)(1) : \chi$ *is a faithful irreducible complex character of* $G\}$;

(2) $q(G) = \min\{m_{\mathbb{Q}}(\chi) c(\chi)(1) : \chi$ *is a faithful irreducible complex character of* $G\}$.

Proof. See [1, Lemma 3.8 and Corollary 3.11] ∎

3. Calculating $p(G)$

First of all, we will state some results that we will need later.

Lemma 3.1. (Zassenhaus-Schur Theorem). *Let $N \lhd G$ and $(|N|, |G/N|) = 1$. Then the sequence*

$$1 \to N \to G \to G/N \to 1$$

splits.

Proof. See [7, Theorem 10.30]. ∎

Lemma 3.2. *Let G be a finite group having an abelian Sylow p-subgroup. Then $|G^{'} \bigcap Z(G)|$ is not divisible by p.*

Proof. See [3, Theorem 5.6]. ∎

Lemma 3.3. *Let $G = \langle a, b : a^m = 1, b^s = a^t, b^{-1}ab = a^r \rangle$. Then*

$$G^{'} = \langle a^{r-1} \rangle.$$

Also $\left| G^{'} \right| = m/(r-1, m)$, where $(r-1, m)$ denotes the greates common divisor of $r-1$ and m.

Proof. See [3, 47.10]. ∎

Let p and q be two different primes. Let G denote a non-abelian metacyclic group, such that, $A = \langle a \rangle$ is a cyclic group of order p^α, $A \lhd G$ and G/A is a cyclic group of order q^β. Then, by Lemma 3.1, $G = A.B$, where $B = \langle b \rangle$ is a cyclic subgroup of G of order q^β and $A \bigcap B = 1$. Hence

$$G = \langle a, b : a^{p^\alpha} = b^{q^\beta} = 1, b^{-1}ab = a^r \rangle \tag{1}$$

for some integer r, such that $(p, r) = 1$. It is easy to prove that

$$b^{-l} a^k b^l = a^{kr^l}. \tag{2}$$

$\sigma(a^i) = b^{-1}a^i b$ is an automorphism of A. Let u be the order of σ. Then $b^u \in Z(G)$. Also $u \mid q^\beta$ and $r^u \equiv 1 \mod p^\alpha$. Therefore $r^{q^\beta} \equiv 1 \mod p^\alpha$. Let $p^\gamma = (p^\alpha, r-1)$. Then $a^{p^{\alpha-\gamma}} \in Z(G)$. So $Z(G) = \langle a^{p^{\alpha-\gamma}} \rangle \langle b^u \rangle$.

A is a unique abelian p-Sylow subgroup of G. B is a q-Sylow subgroup of G and it is abelian, but not a unique q-Sylow subgroup of G, as G is not an abelian group.

Since A is an abelian Sylow p-subgroup of G, p does not divide $|G' \cap Z(G)|$, by Lemma 3.2. So p does not divide $|Z(G)|$. Hence $\gamma = 0$ and

$$(p, r-1) = 1. \tag{3}$$

Therefore, by Lemma 3.3, $G^{'} = A$. Also it is easy to see that,

$$Z(G) = \langle b^u \rangle. \tag{4}$$

Now let $B_G = \bigcap_{g \in G} B^g = 1$ (the core of B in G). Then it is easy to prove that

$$Z(G) = \langle b^u \rangle = 1 \tag{5}$$

and $u = q^\beta$. Hence we have the following lemma.

Lemma 3.4. *Let*

$$G = \langle a, b : a^{p^\alpha} = b^{q^\beta} = 1, a^b = a^r \rangle$$

and $\sigma(a^i) = b^{-1}a^i b$ be an automorphism of A. Then

$$(p, r) = (p, r-1) = 1,$$
$$G' = A \quad and \quad Z(G) = \langle b^u \rangle.$$

Moreover if $B_G = 1$*, then* $Z(G) = 1$.

Lemma 3.5. *Let* G *be as in Lemma* 3.4 *and* $B_G = 1$*. Then*

$$p(G) = p^{\alpha}.$$

Proof. Let $H \leq G$. Then $H = A_1B_1$, where $A_1 \leq A$ and $B_1 \leq B$. Since $B_G = 1$, $(B_1)_G = 1$ and H is not a normal subgroup of G. So the only normal subgroups of G are subgroups of A. This shows that G has a unique minimal normal subgroup of order p. So B is the largest subgroup such that $B_G = 1$. By Lemma 2.1, we have

$$p(G) = p^{\alpha}.$$ ■

Let $K = \{x \in G : x^p = 1\}$. It is easy to prove that $K \leq A$.

Lemma 3.6. *Let* G *be a metacyclic group as in* (1)*. Then* $p(G) \leq p^{\alpha} + q^{\beta}$.

Proof. Since $A \bigcap B = 1$, $A \lhd G$ and A, B are Sylow subgroups of G, the result follows from Theorem 2.2. ■

Let $H \leq G$. Then H is also a metacyclic group and also $H \cap A$ is a normal subgroup of H and $H/(H \cap A) \cong HA/A \leq G/A$ is cyclic. If $H \cap A = 1$, then H is cyclic.

Lemma 3.7. *Let* G *be a metacyclic group. Let* $H \leq G$ *with* $H_G = 1$*, where* H_G *is the core of* H *in* G*. Then* H *is cyclic and* $|H|\ |q^{\beta}$.

Proof. See [2, Lemma 3.6]. ■

Lemma 3.8. *Let* G *be a metacylic group as in* (1) *and* $H \leq G$ *such that* $H \cap A \neq 1$*. Then* $H_G \cap K = K$.

Proof. It is easy. ■

Theorem 3.9 *Let* G *be a metacylic group as in* (1)*. Then*

$$p(G) = \begin{cases} p^{\alpha} & if \quad B_G = 1 \\ p^{\alpha} + q^{\beta} & otherwise. \end{cases}$$

Proof. This follows from Lemmas 3.5, 3.6, 3.7, 3.8. ■

4. Calculating $c(G)$ and $q(G)$

Let G be a group of order pq where p is a prime, q an integer number dividing $p-1$. Then either G is abelian, or

$$G \cong F_{p,q} = \langle a, b : a^p = b^q = 1, b^{-1}ab = a^r \rangle,$$

where r is an element of order q in $\mathbb{Z}_p^*$. (See [6, Proposition 25.7]). These groups are known as Frobenius groups.

Lemma 4.1. *Let p be a prime number, $q \mid p-1$ and $u = (p-1)/q$. Then the group*

$$\begin{aligned} F_{p,q} &= \langle a, b : a^p = b^q = 1, b^{-1}ab = a^r \rangle \\ &= \{a^x b^y : 0 \le x \le p-1, 0 \le y \le q-1\} \end{aligned}$$

has $q + u$ irreducible characters. Of these, q have degree 1 and are given by

$$\chi_n(a^x b^y) = e^{2\pi i n y/q} \quad (0 \le n \le q-1)$$

and u have degree q and are given by

$$\begin{aligned} \phi_j(a^x b^y) &= 0 \quad \textit{if} \quad 1 \le y \le q-1, \\ \phi_j(a^x) &= \sum_{s \in S} \xi^{v_j s x}, \end{aligned}$$

for $1 \le j \le u$, where $\xi = e^{2\pi i/p}$, $v_1 S, \ldots, v_u S$ are the cosets in $\mathbb{Z}^\star{}_p$ of the subgroup S generated by r.

Proof. See [6, Theorem 25.10]. ∎

From now on, let $G = F_{p,q}$. Also note that in Lemma 4.1, q may be any integer number which divides $p - 1$.

It is easy to prove that $Z(G) = 1$. Hence G has a unique minimal normal subgroup. So $p(G) = p$.

Theorem 4.2. *Let $G = F_{p,q}$. Then*

$$p(G) = q(G) = c(G) = p.$$

Moreover the Schur index for all irreducible characters is 1 and non-linear characters are in one Galois orbit and also are faithful.

Proof. Since G has a unique minimal normal subgroup, by Lemma 2.5, G has a faithful character. Let $\chi = \sum_{j=1}^{u} \phi_j$. Then χ is faithful and is also rational valued. Since all non-linear and irreducible characters of G have degree q,

$$\chi(1) = qu = q(p-1)/q = p - 1.$$

Also it is easy to see that, for all $1 \le x \le p-1$, we have

$$\chi(a^x) = \sum_{j=1}^{u} \phi_j(a^x) = -1.$$

This shows that all faithful characters of G is in one Galois orbit and

$$p(G) = q(G) = c(G) = p,$$

also the Schur index is 1. ■

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