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Generalizations on Common Fixed Points for Three Commuting Mappings in Metric and Menger Spaces*

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Abstract. We prove common fixed point theorems for three commuting self-mappings on complete metric spaces which generalize and unify some previously known results. Examples and applications to Menger spaces and random operator equations are also given.

1. Introduction

In [4] we established common fixed point theorems for a pair of commuting self-mappings on a complete metric space satisfying the g-quasi-contraction and a metric condition of Fisher-Sessa, or Fisher-Iseki type, where g is a self-function of $\mathbb{R}^+$ satisfying the following properties

$(g1)$ g is a non-decreasing function;
$(g2)$ g is right-continuous;
$(g3)$ $\forall t > 0 \quad g(t) < t$;
$(g4)$ $\exists \lim\limits_{t\to\infty} \dfrac{g(t)}{t} < 1.$

Based on the approaches of [4, 8], we extended the results of [4] to the case of three commuting mappings in [1, 3]. Afterwards following a suggestion by Prof. Tien, we made an attempt in order to improve the mentioned results by considering instead of $(g4)$ the following weaker condition

$(\overline{g4})$ $\exists \overline{\lim\limits_{t\to\infty}} \dfrac{g(t)}{t} < 1.$

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In fact it turned out that one can slightly adapt the arguments of [1, 3] for the purpose above. With condition $(\overline{g4})$ involved, the main goal of the present paper is to show that the results of [1, 3] remain true for a wider class of functions g. As the reader will see later, the heart of the paper is Lemma 5.2 below, which shows that any self-function of $\mathbb{R}^+$, satisfying properties $(g1)-(g3)$, $(\overline{g4})$, can be majorated by a nice function satisfying $(g1)-(g4)$, *i.e.* the described situation can be reduced in the obvious manner to the one treated in [1, 3]. However in order to underline several interesting features of $(\overline{g4})$ one feels plausible an exposition with proofs based generically on exploiting the existence of upper limit in $(\overline{g4})$ that is certainly of independent interest. In a sense the paper should be considered as an extended version of preprint [3]. As for illustration purpose we discuss several examples in detail. We give also some applications to the class of Menger spaces with t-norm $T \geq T_m$ and the theory of random operator equations.

2. Preliminaries

Let (X, d) be a complete metric space, $f_i,\ i = 0, 1, 2$ three commuting self-mappings on X such that:
(1) $f_i(X) \subset f_0(X), \quad i = 1, 2;$
(2) f_1 and f_2 satisfy the following g-quasi-contractive condition

$$d(f_1x, f_2y) \leq g\big(m_0(f_1x, f_2y)\big), \quad \forall x, y \in X, \tag{2.1}$$

where $m_0(f_1x, f_2y) := \max\ \{d(f_0x, f_0y), d(f_0x, f_1x), d(f_0y, f_2y), d(f_0x, f_2y), d(f_0y, f_1x)\}$ and g is a function $: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying properties $(g1)-(g3)$, $(\overline{g4})$.

Let us introduce the following new conditions which are generalizations of metric conditions of Fisher–Sessa type, or Fisher-Iseki type (cf. [9, 10] and also [4]):
There exists a point $x \in X$ such that

$$\sup_{y,y' \in \mathcal{O}_{f_0}(x)} \big\{d\big(f_i^{n+1}y, f_i^n y'\big), \quad n = 0, 1, 2, \ldots\ ; \quad i = 1, 2\big\} < \infty. \tag{2.2}$$

There exist a point $x \in X$ and a constant M such that

$$d(f_i^{n+1}y, f_i^n y') \leq (n+1)M, \quad \forall y, y' \in \mathcal{O}_{f_0}(x), \tag{2.3}$$

for $n = 0, 1, 2, \ldots$ and $i = 1, 2$.

Here $\mathcal{O}_{f_0}(x)$ denotes the orbit of x under f_0. It should be noted that in some cases it is worth considering also the following conditions for the function g:
$(g1')$ $g(0) = 0;$
$(g2')$ g is upper semi-continuous.
Before proceeding further we make some easy remarks: $(g1)$ and $(g3)$ clearly imply (g1'); $(g1)$ and $(g2')$ imply $(g2)$.

The following claim compares $(\overline{g4})$ with property $(g4')$ first appeared in [5]
$(g4')$ $\lim_{t \to \infty}(t - g(t)) = \infty,$

Claim. *With property* $(g3)$ *fulfilled we have implication* $(\overline{g4}) \Longrightarrow (g4')$*, while the converse* $(g4') \Longrightarrow (\overline{g4})$ *is not true.*

Proof. Assume $\overline{q} := \overline{\lim_{t\to\infty}} \frac{g(t)}{t} < 1$. Taking $\overline{q}_0$ such that $\overline{q} < \overline{q}_0 < 1$, then by definition of the upper limit one sees that there exists t_0 (depending on $\overline{q}_0$) such that $g(t)/t \notin [\overline{q}_0, 1]$ for all $t \geq t_0$, or equivalently

$$\frac{g(t)}{t} < \overline{q}_0, \quad \forall\, t \geq t_0.$$

Hence

$$t - g(t) > t(1 - \overline{q}_0) \to +\infty, \quad \text{as} \quad t \to +\infty,$$

as claimed.

For the second statement one can take $g_\alpha(t) := \alpha t$, if $t \in [0, 1)$, and $g_\alpha(t) := [t]\sqrt{t}/(\sqrt{t} + 1)$, if $t \in [1, +\infty)$, where $\alpha \in [0, 1/2]$ is a constant and $[t]$ denotes the greatest integer not exceeding t; another example is $g(t) := 0$, if $t \in [0, 3)$, and $g(t) := [t] - [\log_2 t] - 1$, if $t \in [3, +\infty)$.

For our later use we need the following two auxiliary lemmas.

Lemma 2.1. *Assume that we are given* g *satisfying properties* $(g1')$, $(g2')$, $(g3)$. *Then*

(i) *for every* $t > 0$ *fixed one has* $\lim_{n\to\infty} g^n(t) = 0$;

(ii) *for any sequence* $\{d_n\}$ *of non-negative real numbers such that* $d_{n+1} \leq g(d_n)$, $n = 1, 2, \dots$, *we have* $\lim_{n\to\infty} d_n = 0$.

For a proof see, e.g. [4].

Lemma 2.2. *Assume that the conditions* $(g1), (g3), (\overline{g4})$ *are fulfilled. Then*

(i) *For any fixed integer* $k > 0$

$$\overline{\lim_{t\to\infty}} \frac{g^k(t)}{t} \leq \overline{q}^k;$$

where $\overline{q} := \overline{\lim_{t\to\infty}}\, g(t)/t$.

(ii) *There exist a positive integer* k_0 *and* $t_0 := t(k_0)$ *depending on* k_0 *such that*

$$g^k(t) < \frac{t}{2}, \quad \forall t \geq t_0, \quad k \geq k_0.$$

Proof.

(i) If $g(t)$ is bounded, as $t \to \infty$, (*i.e.* $\overline{q} = 0$), then the first statement is obvious. Otherwise one may apply the product formula for upper limits to

$$g^k(t)/t = [g^k(t)/g^{k-1}(t)].[g^{k-1}(t)/g^{k-2}(t)] \dots [g^2(t)/g(t)].[g(t)/t].$$

(ii) As above after taking $\overline{q}_0$: $\overline{q} < \overline{q}_0 < 1$ one can choose k_0 satisfying $\overline{q}_0^{k_0} < 1/2$ and t_0, depending on $k_0, \overline{q}_0$ such that in view of $(g3)$, (i) and by definition of the upper limit we have for any $\forall k \geq k_0, \quad t \geq t_0$

$$\frac{g^k(t)}{t} \le \frac{g^{k_0}(t)}{t} \le \overline{q}^{k_0} < \overline{q}_0^{k_0} < \frac{1}{2},$$

hence the conclusion (ii) easily follows. ■

Note that examples above show that conclusion (ii) of Lemma 2.2 is false if $\overline{q} = 1$. In fact there are examples satisfying $(g1) - (g3), (\overline{g4})$, but $\lim_{t\to\infty} g(t)/t$ does not exist. For every positive $\varepsilon_0 < 1/2$ fixed we construct the sequence $\{n_i\}$ and function $g_{\varepsilon_0}(t)$ as follows: $n_0 = 0,\ n_1 = 1,\ n_{2k} = n_{2k-1} + 1,\ n_{2k+1} = [\delta n_{2k}] + 1,\ k = 1, 2, \dots$, where $\delta := \dfrac{1/2 + \varepsilon_0}{1/2 - \varepsilon_0}$. We put $g_{\varepsilon_0}(t) := n_{2k-1}\big(1/2 - \varepsilon_0\big)$ for $t \in [n_{2k-1}, n_{2k})$ and $g_{\varepsilon_0}(t) := n_{2k}\big(1/2 + \varepsilon_0\big)$ for $t \in [n_{2k}, n_{2k+1})$. It is easy to verify that $g_{\varepsilon_0}(t)$ satisfies $(g1) - (g3), (\overline{g4})$: $\overline{\lim_{t\to\infty}} g(t)/t = 1/2 + \varepsilon_0$. At the same time $g_{\varepsilon_0}(n_{2k-1})/n_{2k-1} \to 1/2 - \varepsilon_0,\ g_{\varepsilon_0}(n_{2k})/n_{2k} \to 1/2 + \varepsilon_0$ as $k \to \infty$.

Let $\delta(A) := \sup\{d(x, y) : x, y \in A\}$ denote the diameter of a subset A of X and

$$\mathcal{O}(x, \infty) := \big\{f_0^k f_1^m f_2^n x : k, m, n = 0, 1, 2, \dots\big\};$$

Lemma 2.3. *Let commuting mappings* $f_0, f_1,\ f_2$ *satisfy* $(2.1) - (2.2)$. *Then*

$$\delta[\mathcal{O}(x, \infty)] < \infty.$$

Proof. If we denote by $\mathcal{O}(x, N) := \{f_0^k f_1^m f_2^n x : 0 \le k, m, n \le N\},\quad N = 1, 2, \dots$, then $\delta[\mathcal{O}(x, 1)] \le \delta[\mathcal{O}(x, 2)] \le \dots$, so it is clear that $\delta[\mathcal{O}(x, \infty)] = \sup\big\{\delta[\mathcal{O}(x, N)] : N = 1, 2, \dots\big\}$. Next putting $x_{k,m,n} := f_0^k f_1^m f_2^n x$, in view of (2.1) one has for $0 \le k, m, n, k_1, m_1, n_1 \le N,\ \ m > 0,\ n_1 > 0$

$$\begin{aligned} d(x_{k,m,n}, x_{k_1,m_1,n_1}) &\le g\big(\delta\{x_{k+1,m-1,n}, x_{k_1+1,m_1,n_1-1}, x_{k,m,n}, x_{k_1,m_1,n_1}\}\big) \\ &\le g\big(\delta[\mathcal{O}(x, N)]\big). \end{aligned}$$

Since f_0, f_1, f_2 are commuting and because of property $(g3)$ of g without loss of generality we may assume that there are the following possibilities we have to consider:

1. $\delta[\mathcal{O}(x, N)] = d(f_0^k x, f_0^{k_1} x)$. In view of (2.2) and the triangle inequality we have

$$\delta[\mathcal{O}(x, N)] \le d(f_0^k x, f_1 f_0^k x) + d(f_1 f_0^k x, f_0^{k_1} x) \le 2L,$$

where

$$L := \sup_{y, y' \in \mathcal{O}_{f_0}(x)} \big\{d(f_i^{n+1} y, f_i^n y'),\quad n = 0, 1, 2, \dots;\quad i = 1, 2\big\} < \infty.$$

2. $\delta[\mathcal{O}(x, N)] = d(f_0^k x, x_{k_1,m_1,n_1})$ with $m_1 > 0$. Applying (2.1), (2.2) and the triangle inequality one gets

$$\delta[\mathcal{O}(x,N)] \le d(f_0^k x, f_2 f_0^k x) + d(f_2 f_0^k x, f_1^{m_1} f_2^{n_1} f_0^{k_1} x) \le L + g\big(\delta[\mathcal{O}(x,N)]\big). \quad (2.4)$$

3. $\delta[\mathcal{O}(x,N)] = d(f_2^n f_0^k x, f_2^{n_1} f_0^{k_1} x)$ with $n > n_1$. The case $n_1 = 0$ reduces to the inequality (2.4) above. Further again the triangle inequality implies that

$$\delta[\mathcal{O}(x,N)] \le d(f_2^n f_0^k x, f_1^n f_0^k x) + d(f_1^n f_0^k x, f_2^{n_1} f_0^{k_1} x).$$

So we can apply (2.1) to both terms on the right-hand side, e.g. for $d(f_2^n f_0^k x, f_1^n f_0^k x)$, say, we have

$$d(f_2^n f_0^k x, f_1^n f_0^k x) \le g\big(\delta\{f_2^{n-1} f_0^{k+1} x, f_1^{n-1} f_0^{k+1} x, f_2^n f_0^k x, f_1^n f_0^k x\}\big).$$

In view of $(g1), (g3)$ and (2.2), either $d(f_2^n f_0^k x, f_1^n f_0^k x) \le g(L) \le L$, or one can descend further by applying (2.1)

$$\delta[\mathcal{O}(x,N)] \le g^n\big(\delta[\mathcal{O}(x,n)]\big) + g^{n_1}\big(\delta[\mathcal{O}(x,n)]\big) \le 2g^{n_1}\big(\delta[\mathcal{O}(x,N)]\big). \quad (2.5)$$

Now if we choose n_0 such that $\overline{q}_0^{n_0} < 1/2$, then Lemma 2.2 shows that (2.5) is impossible as far as $n_1 \ge n_0$ and $\delta[\mathcal{O}(x,N)]$ is unbounded for N sufficiently large. Otherwise we have

$$\delta[\mathcal{O}(x,N)] \le \delta[\mathcal{O}(x,n_0)] + g\big(\delta[\mathcal{O}(x,n)]\big).$$

One concludes therefore that either $\delta[\mathcal{O}(x,N)]$ is bounded for all N, or there exists a sequence $\{N_n\}$ such that $N_n \to \infty$ as $n \to \infty$ and $\forall N_n$: (2.4) (or (2.5)) holds. The second possibility leads us to a contradiction with $(g4')$. ∎

3. Main Results

The aim of this section is to prove our main theorems which are generalizations of the results of [1, 3, 4, 9, 10] (cf. also references therein).

Theorem 3.1. *Let (X,d) be a complete metric space; f_i $i = 0,1,2$ commuting self-mappings of X such that*

(i) f_i, $i = 0,1,2$ *satisfy conditions* $(2.1) - (2.2)$ *for a function* g *with properties* $(g1) - (g3), (\overline{g4})$;

(ii) $f_j(X) \subset f_0(X)$, $j = 1,2$;

(iii) f_0 *is continuous.*

Then there exists a unique common fixed point in X for f_0, f_1, f_2.

Proof.
1. Let us construct the sequence $\{x_n\}$ as follows: for x_0 arbitrary in X, let $x_1 \in X$, guaranteed by (ii), be such that $f_2 x_0 = f_0 x_1$ and denote it by y_0. Having defined $x_n \in X$, let $x_{n+1} \in X$ be such that $f_1 x_{2k+1} = f_0 x_{2k+2}$ with $n = 2k+1$, and $f_2 x_{2k} = f_0 x_{2k+1}$ with $n = 2k$. Letting $y_{2k+1} = f_1 x_{2k+1}$, and $y_{2k} = f_2 x_{2k}$ we prove that $d(f_0 y_n, f_0 y_{n+1}) \to 0$ as $n \to \infty$. Say for $n = 2k$ we have by (2.2)

$$d(f_0 y_{2k}, f_0 y_{2k+1}) \le g\big(\delta\{f_0 y_{2k-1},\ f_0 y_{2k},\ f_0 y_{2k+1}\}\big).$$

So either $d(f_0 y_{2k}, f_0 y_{2k+1}) \le g\big(d(f_0 y_{2k-1}, f_0 y_{2k})\big)$, or descending further as in the proof of Lemma 2.3 yields

$$d(f_0y_{2k-1}, f_0y_{2k+1}) \leq g^{k-1}\big(\delta[\mathcal{O}(x,\infty)]\big).$$

Both cases together with Lemma 2.3 imply $d(f_0y_{2k}, f_0y_{2k+1}) \leq g^k\big(\delta[\mathcal{O}(x,\infty)]\big)$. Analogously for $n = 2k+1$ $d(f_0y_{2k+1}, f_0y_{2k+2}) \leq g^{k+1}\big(\delta[\mathcal{O}(x,\infty)]\big)$. Summing up one concludes in view of Lemmas 2.2 and 2.3 that in either case $\lim\limits_{n\to\infty} d(f_0y_n, f_0y_{n+1}) = 0$.

2. Now we prove that the sequence $\{f_0y_n\}$ is a Cauchy sequence. We assume contrariwise that this sequence is not Cauchy's, that is

$(*)$ $\exists\, \varepsilon > 0$, for every integer N one can choose $n(N) > N$ such that

$$d(f_0y_{n(N)}, f_0y_N) \geq 2\varepsilon.$$

Since $d(f_0y_n, f_0y_{n+1}) \to 0$ as $n \to \infty$, there exists $N(\varepsilon)$ such that if $n > N(\varepsilon)$ then $d(f_0y_n, f_0y_{n+1}) < \varepsilon$. In particular it is easy to see that for any $N = 2k > N(\varepsilon)$ one can choose $2m(k) > 2k$ such that

$$d(f_0y_{2k}, f_0y_{2m(k)}) > \varepsilon. \tag{3.1}$$

(Indeed if $n(2k)$ chosen in (*) is odd, say $n(2k) = 2m(k)+1$ one can write

$$d(f_0y_{2m(k)}, f_0y_{2m(k)+1}) + d(f_0y_{2k}, f_0y_{2m(k)}) \geq d(f_0y_{2k}, f_0y_{2m(k)+1}) \geq 2\varepsilon,$$

which implies (3.1), since $d(f_0y_{2m(k)}, f_0y_{2m(k)+1}) < \varepsilon$). One can also assume that $2m(k)$ is chosen minimum among such even integers $> 2k$ so that $d(f_0y_{2k}, f_0y_{2m(k)-2}) \leq \varepsilon$. Combining this and (3.1) we get

$$\varepsilon < d(f_0y_{2k}, f_0y_{2m(k)}) \leq \varepsilon + d(f_0y_{2m(k)-2}, f_0y_{2m(k)-1}) + d(f_0y_{2m(k)-1}, f_0y_{2m(k)}),$$

or as k tends to ∞

$$\lim_{k\to\infty} d(f_0y_{2k}, f_0y_{2m(k)}) = \varepsilon. \tag{3.2}$$

On the other hand we have

$$d(f_0y_{2k}, f_0y_{2m(k)}) \leq d(f_0y_{2k}, f_0y_{2k+1}) + d(f_0y_{2k+1}, f_0y_{2m(k)}). \tag{3.3}$$

One has to estimate the term $d(f_0y_{2k+1}, f_0y_{2m(k)})$ from the right-hand side of (3.3). From the above and the condition (2.1) of the theorem it follows that

$$\begin{aligned}
&d(f_0y_{2k+1}, f_0y_{2m(k)}) \leq g\big(\max\{\varepsilon + d(f_0y_{2m(k)-2}, f_0y_{2m(k)-1}), d(f_0y_{2k}, f_0y_{2k+1}),\\
&d(f_0y_{2m(k)-1}, f_0y_{2m(k)}), \varepsilon + d(f_0y_{2m(k)-2}, f_0y_{2m(k)-1}) + d(f_0y_{2m(k)-1}, f_0y_{2m(k)}),\\
&\varepsilon + d(f_0y_{2m(k)-2}, f_0(y_{2m(k)-1})) + 2d(f_0y_{2m(k)-1}, f_0y_{2m(k)}) + d(f_0y_{2k}, f_0y_{2k+1})\}\big).
\end{aligned}$$

Since $\lim\limits_{n\to\infty} d(f_0y_n, f_0y_{n+1}) = 0$ and because of (g1) for g this yields

$$d(f_0y_{2k+1}, f_0y_{2m(k)}) \leq g(\varepsilon + \delta) \tag{3.4}$$

for $\delta > 0$ and k sufficiently large. In view of (3.2)–(3.4) and the right-continuity of g one obtains as $k \to \infty$ and $\delta \to 0+$ $\varepsilon \leq g(\varepsilon)$, that contradicts $(g3)$. We

have proven that the sequence $\{f_0(y_n)\}$ is Cauchy's, hence convergent. Let us call the limit by u.
3. Now we show that u is a common fixed point of f_0, f_1, f_2 and this common fixed point is unique. We first claim that $u = f_0u$ Indeed, if $u \neq f_0u$ then from the condition (2.1) and properties of f_0 we have

$$d(f_0y_{2k+1}, f_0^2y_{2k+2}) = d(f_1y_{2k}, f_2f_1y_{2k})$$
$$\leq g\big(\delta\big\{f_0y_{2k}, f_0f_1y_{2k}, f_1y_{2k}, f_0^2y_{2k+2}\big\}\big) \leq g\big(\delta\big\{u, f_0u, u, f_0u\big\}\big)$$

as k is sufficiently large. Letting k tend to ∞ gives a contradiction with $(g3)$. So that $d(u, f_0u) = 0$, in other words u is a fixed point of f_0. Using the condition (2.1) of the theorem and assume that $u \neq f_1u$ we estimate

$$d(f_1u, f_0y_{2k}) = d(f_1u, f_2y_{2k-1}) \leq g\big(\delta\big\{f_0u, f_0y_{2k-1}, f_2y_{2k},\ f_1u)\big\}\big) \leq g\big(d(u, f_1u)\big)$$

as k sufficiently large. Letting k tend to ∞ gives a contradiction with $(g3)$. So that u is a fixed point of f_1. Analogously putting $x = y = u$ in the condition (2.1) of the theorem one obtains that u is also a fixed point of f_2. The uniqueness is almost evident. ∎

Corollary 3.2. *Let (X, d) be a complete metric space; f_i $i = 1, 2$ commuting self-mappings of X satisfying conditions* (2.1) *with f_0 =id and* (2.2)*. Then there exists a unique common fixed point in X for f_1, f_2.*

Theorem 3.3. *Let (X, d) be a complete metric space; f_i, $i = 0, 1, 2$ commuting self-mappings of X such that*
(i) *f_i, $i = 0, 1, 2$ satisfy conditions* (2.1), (2.3) *for a function g with properties* $(g1) - (g3), (\overline{g4})$*;*
(ii) $f_j(X) \subset f_0(X)$, $j = 1, 2$;
(iii) *f_0 is continuous.*
Then there exists a unique common fixed point in X for f_0, f_1, f_2.

Lemma 3.4. *Under the hypotheses of Theorem* 3.3 *there exists a constant L such that for $N = 1, 2, \ldots$.*

$$\delta[\mathcal{O}(x, N)] \leq 2NL.$$

Proof. Indeed as one has seen in the proof of Lemma 2.3, the only step we have to replace (2.2) with (2.3) is step 2. We have either

$$\delta[\mathcal{O}(x, N)] \leq 2nM \leq 2NM,$$

where M is the constant from (2.3), or by descending argument $\delta[\mathcal{O}(x, N)]$ are bounded. Thus the conclusion of the lemma follows.

Before proving Theorem 3.3 one notes that properties $(g1)$ and $(\overline{g4})$ imply the following property

$(g5)$ $$\lim_{k\to\infty} g^k(ka) = 0 \text{ for } a > 0 \text{ fixed.}$$

Indeed recall that $\overline{q} := \overline{\lim_{t\to\infty}}\, g(t)/t < 1$ in view of $(\overline{g4})$, for a chosen $\overline{q} < \overline{q}_0 < 1$, there exists t_0 as above so that

$$\frac{g(t)}{t} < \overline{q}_0, \quad \forall t \geq t_0,$$

in particular for any $t \geq t_0$: $g(t) < \overline{q}_0 t$. Clearly by $(g1)$ and induction applied to this inequality one gets: $g^k(t) < \overline{q}_0^k t$. Furthermore putting $t = ka$ we have for k sufficiently large: $g^k(ka) < \overline{q}_0^k ka$. Thus as k tends to ∞ one obtains the desired $(g5)$.

We continue the proof of Theorem 3.3. Let $\{x_n\}$ be the sequence constructed in proving Theorem 3.1. Taking into account Lemma 3.4 we have

$$d(f_0 y_{2k+1}, f_0 y_{2k+2}) \leq g^{k+1}\big(\delta[\mathcal{O}(x, 2k+2)]\big) \leq g^{k+1}\big(4(k+1)L\big),$$
$$d(f_0 y_{2k}, f_0 y_{2k+1}) \leq g^{k}\big(\delta[\mathcal{O}(x, 2k+1)]\big) \leq g^{k}\big((4k+2)L\big).$$

So property $(g5)$ yields $\lim_{k\to\infty} d(f_0 y_n, f_0 y_{n+1}) = 0$. The rest of the proof is identical to steps 2 and 3 in the proof of Theorem 3.1. ∎

Corollary 3.5. *Let (X, d) be a complete metric space; let f_i $i = 1, 2$, be commuting self-mappings of X satisfying condition* (2.1) *with $f_0 =$*id *and* (2.3). *Then there exists a unique common fixed point in X for f_1, f_2.*

Note that the main results of [4] (and in particular of [5, 7, 9, 10]) are immediate consequences of Corollaries 3.2, 3.5.

4. Examples

4.1. We can give various examples satisfying the conditions of Theorems 3.1 – 3.3 above. Let $X = [0, +\infty)$ with usual metric. Consider the following self-mappings $f_0(x) = q_0 x$, $f_1(x) = q_1 x$ with $q_0 > q_1 > 0$, and $f_2(x) \equiv 0$. One checks easily that f_0, f_1, f_2 satisfy the conditions of Theorems 3.1, 3.3 above with function $g(t) = qt$, $q := q_1/q_0$. Hence they have a unique common fixed point in X.

4.2. Let $\mathbb{N}$ be the set of positive integers. Consider the following self-mappings of $\mathbb{N}$: $f_0 = \mathrm{id}$, $f_1(n) := n+1$, $f_2(n) := n+2$. Clearly f_0, f_1 and f_2 are commuting. Note that any metric d on $\mathbb{N}$ should satisfy $d(n, n) = 0$. One may try to choose a metric d such that $d(m, n) \downarrow t_0 \geq 0$ as $\min\{m \neq n\} \to \infty$. Certainly a plenty of such metrics exists, for instance, one can define for $m > n : d(m, n) = d(n, m) := a + 1/n^\alpha$ with non-negative a, α; or $d(m, n) = d(n, m) := a + 1/\zeta_n(1)$ with non-negative a; or $d(m, n) = d(n, m) := a + 1/\zeta_n(2)$ with $a \geq -6/\pi^2$, where for two last families we use the notation $\zeta_n(s) := 1 + 1/2^s + \cdots + 1/n^s$ - the "truncated" Riemann zeta-function. With these circumstances and if $t_0 > 0$ $\mathbb{N}$ is a complete metric space. In fact f_0, f_1 and f_2 would satisfy the conditions (2.1)-(2.2) for a chosen function g with properties $(g1) - (g3), (g4')$. But one could not have a choice for g to satisfy property $(\overline{g4})$: such a function fails to satisfy $(g3)$ at $t = t_0$.

These examples show that the conditions $(g2), (g3)$ and $(\overline{g4})$ are essential. The only way to get rid of this "difficulty" at $t = t_0$ is as follows: one has to choose the metric d such that $t_0 = 0$, *e.g.* as $a = 0, \alpha > 0$; $a = 0$; $a = -6/\pi^2$ respectively in the above families of metrics. But in this case $\mathbb{N}$ is not complete. Clearly one obtains a completion by adding the point ∞ to $\mathbb{N}$ with natural ordering $n < \infty$, for all $n \in \mathbb{N}$ and f_1, f_2 are extended well to the whole $\mathbb{N} \cup \{\infty\}$ in the obvious manner. So that ∞ is the unique fixed point of f_0, f_1, f_2 in accordance with Theorem 3.1.

5. Applications

We now proceed to the case of probabilistic (random) metric spaces. First let us mention some definitions [11 - 13]. Let δ_0 denote the set of all distribution functions F with $F(0) = 0$ (F is nondecreasing, left-continuous and $\sup_{t\in\mathbb{R}} F(t) = 1$). A probabilistic metric space (a PM-space) is an ordered pair $(X, \mathcal{F})$ consisting of a nonempty set X and a symmetric mapping $\mathcal{F}: X \times X \to \delta_0$ ($\mathcal{F}(x, y)$ is denoted by $F_{x,y}$ for $(x, y) \in X$) which satisfies the following conditions:
(1) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$.
(2) If $F_{x,z}(t) = 1$ and $F_{z,y}(s) = 1$, then $F_{x,y}(t + s) = 1$ for all $x, y, z \in X$ and $t, s > 0$.

A Menger space is a triplet $(X, \mathcal{F}, T)$, where $(X, \mathcal{F})$ is a PM-space, T is a triangular norm (t-norm) and the Menger triangular inequality $F_{x,y}(t + s) \geq T\big(F_{x,z}(t), F_{z,y}(s)\big)$ holds for all $x, y, z \in X$ and $t, s > 0$. Recall that a t-norm T is a commutative, associative and nondecreasing mapping $T: [0, 1] \times [0, 1] \to [0, 1]$ such that $T(0, 0) = 0$, $T(a, 1) = a$. There are two important t-norms: $T(a, b) := \min\,(a, b)$ and $T_m(a, b) := \max\,(a + b - 1, 0)$ which will be used frequently in the sequel. The case $(X, \mathcal{F}, \min)$ was studied extensively (see, e.g. [6, 14] and cited references therein). In this case for each $\lambda \in (0, 1)$ one can define a pseudo-metric d_λ by putting $d_\lambda(x, y) = \sup\,\{t: F_{x,y}(t) \leq 1 - \lambda\}$ so that

$$F_{x,y}(t) > 1 - \lambda \quad \text{if and only if} \quad t > d_\lambda(x, y). \tag{5.1}$$

The following lemma is a key point in applications below.

Lemma 5.1. *Let $g: \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying $(g1) - (g3), (\overline{g4})$. Then there exists a continuous and strictly increasing function $f: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $(g4)$ such that $g(t) \leq f(t) < t$ for all $t > 0$.*

Proof. Firstly one constructs a continuous and strictly increasing function f_0 on $[0, 1]$ such that $g(t) \leq f_0(t) < t$ for $t > 0$, as in [6, Proposition 1]

$$a_0 := \min_{[1,2]}(t - g(t)), \quad a_n := \min_{[1/(n+1),1/n]}(t - g(t)) \quad \text{for} \quad n \in \mathbb{N},$$
$$b_1 := \min\,\{a_0, a_1\}, \quad b_n := \min\,\big\{a_0, \cdots, a_n;\ 1/n(n+1)\big\}, \quad \text{for} \quad n \geq 2,$$
$$f_0(0) := 0, \quad f_0(1/n) := 1/n - b_n, \quad \text{and for} \quad t \in [0, 1/n(n+1)]$$
$$f_0(1/(n+1) + t) := [1 - n(n+1)t]f(1/(n+1)) + n(n+1)tf(1/n).$$

Next as in the proof of Claim and Lemma 2.2, for a chosen $\overline{q} < \overline{q}_0 < 1$, in view of $(\overline{g4})$, there is $N_0 = N(\overline{q}_0)$ such that for all $t > N_0$, $g(t)/t < \overline{q}_0$. By $(g1)-(g3)$ one can take a partition $[1, N_0] = \bigcup_{i=1}^{k} I_i$ of $[1, N_0]$ by closed subintervals I_i such that $g(t)/t$ is continuous on I_i for $i = 1, ..., k$, and putting

$$q_i := \max_{I_i} g(t)/t,\ \overline{q}_1 := \max_{0\le i\le k} \{f_0(1), q_i\} < 1,\ t_0 := \sup\{t \in [0,1] : f_0(t) = \overline{q}_1 t\},$$

one obtains the required function $f(t) := f_0(t)$, if $t \in [0, t_0]$, and $f(t) := \overline{q}_1 t$, if $t \in [t_0, +\infty)$. ∎

Remark. There are examples showing the necessity of involving the construction of $f_0(t)$ on $[0, 1]$; for instance, one can take $g(t) := 1 - e^{-t}$, if $t \in [0, 1)$, and $g(t) := 2t/3$, if $t \in [1, +\infty)$. At this point we may see another interesting feature of the condition $(g4)$.

One can have an easy application of the result above to Menger spaces with $T = \min$. Recall that the (ε, λ)-topology in a Menger space $(X, \mathcal{F}, T)$ can be defined by the family $\{U_x(\varepsilon, \lambda); x \in X, \varepsilon > 0, \lambda \in (0, 1)\}$ of (ε, λ)-neighborhoods, where

$$U_x(\varepsilon, \lambda) := \{y \in X;\quad F_{x,y}(\varepsilon) > 1 - \lambda\}.$$

If $\sup_{a\in(0,1)} T(a, a) = 1$ then $(X, \mathcal{F}, T)$ is a Hausdorff topological space in the (ε, λ)-topology. It is easy to see that $d_\lambda(x, y) = \inf\{\varepsilon > 0:\ y \in U_x(\varepsilon, \lambda)\}$ in the case $T = \min$. The family $\{d_\lambda\}$ generates the same topology in $(X, \mathcal{F}, \min)$. In particular it satisfies the following property: $d_\lambda(x, y) = 0$, $\forall\lambda \in (0, 1)$ if and only if $x = y$. We now formulate a probabilistic version of conditions (2.2), (2.3): There exist a point $x \in X$ and a constant L such that for all $y, y' \in \mathcal{O}_{f_0}(x)$

$$F_{f_i^{n+1}y, f_i^n y'}(L) = 1, \quad n = 0, 1, 2, \dots;\ \ i = 1, 2. \tag{5.2}$$

There exist a point $x \in X$ and a bounded function $\varphi: (0, 1) \to \mathbb{R}^+$ such that

$$\inf_{y,y'\in\mathcal{O}_{f_0}(x)} \{F_{f_i^{n+1}y, f_i^n y'}((n+1)\varphi_\lambda), \quad i = 1, 2\} > 1 - \lambda, \quad \forall\ \lambda \in (0, 1), \tag{5.3}$$

where $\varphi_\lambda := \varphi(\lambda)$.

As an immediate consequence of Theorems 3.1 and 3.3 one obtains

Corollary 5.2. *Let $(X, \mathcal{F}, min)$ be a complete Menger space, f_0, f_1, f_2 three commuting self-mappings with f_0 continuous and $f_i(X) \subset f_0(X)$ for $i = 1, 2$. Assume that there exists a function $g: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $(g1)-(g3), (\overline{g4})$ such that for all x, y in X and $t > 0$*

$$F_{f_1x, f_2y}(g(t)) \ge \min\{F_{f_0x, f_0y}(t), F_{f_0x, f_1x}(t), F_{f_0y, f_2y}(t), F_{f_0x, f_2y}(t), F_{f_0y, f_1x}(t)\}. \tag{5.4}$$

If, in addition, either (5.2)*, or* (5.3) *holds for some* $x \in X$*, then there exists a unique common fixed point in* X *for* f_0, f_1, f_2*.*

Let $(X, \mathcal{F}, T)$ be a Menger space. It is well known that if t-norm T satisfies $\sup_{a\in(0,1)} T(a,\ a) = 1$, then in the (ε, λ)-topology X is a metrizable topological space. A t-norm T_1 is stronger than a t-norm T_2 (written as $T_1 \geq T_2$) if $T_1(a, b) \geq T_2(a, b),\ \forall a, b \in [0, 1]$. Moreover if there is a pair (a, b) with strict inequality, then we say T_1 strictly stronger than T_2. We now extend the method here to the class of Menger spaces with t-norm $T \geq T_m$, and since by [13] every E-space is a Menger space w.r.t. t-norm T_m, we can apply the results of this type to the theory of random operator equations. In the case $T \geq T_m$ one can use the following metric with nice properties (cf. [2])

$$\beta(x, y) := \inf\{u:\quad F_{x,y}(u^+) > 1 - u\}.$$

Theorem 5.3. *Let* $(X, \mathcal{F}, T)$ *be a complete Menger space with* $T \geq T_m$*,* f_0, f_1, f_2 *commuting mappings of* X *into itself with* f_0 *continuous and* $f_i(X) \subset f_0(X)$ *for* $i = 1, 2$*. Assume that there exists a function* $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ *satisfying* $(g1) - (g3), (\overline{g4})$ *such that for all* x, y *in* X *and* $t > 0$

$$\begin{aligned} &1 - F_{f_1x,f_2y}(g(t)) \leq g\big(1 - \min\big\{F_{f_0x,f_0y}(t), F_{f_0x,f_1x}(t), \\ &F_{f_0y,f_2y}(t), F_{f_0x,f_2y}(t), F_{f_0y,f_1x}(t)\big\}\big). \end{aligned} \tag{5.5}$$

Then f_0, f_1, f_2 *have a unique common fixed point in* X*.*

Proof. By Lemma 5.1 there exists a continuous and strictly increasing (hence invertible) self-function of $\mathbb{R}^+$ satisfying $(g4)$ such that $g(t) \leq f(t) < t,\ \forall\, t > 0$. Since β is bounded we now show that the condition (2.1) of Theorems 3.1, 3.3 holds w.r.t. the metric β. Assume the contrary that there exist x, y in X such that

$$\beta(f_1x, f_2y) > f\big(m_0(f_1x, f_2y)\big), \quad \text{i.e.} \quad t := f^{-1}\big(\beta(f_1x, f_2y)\big) > m_0(f_1x, f_2y),$$

here $m_0(f_1x, f_2y)$ is defined as in (2.1) w.r.t. the metric β. So in view of the properties of the metric β, and by using the monotonicity of f and distribution functions we have

$$\begin{aligned} &1 - F_{f_1x,f_2y}\big(g(t)\big) \geq 1 - F_{f_1x,f_2y}\big(f(t)\big) \geq \beta(f_1x, f_2y) > f\big(m_0(f_1x, f_2y)\big) \\ &\geq f\big(\max\big\{1 - F_{f_0x,f_0y}(\beta(f_0x, f_0y)^+), 1 - F_{f_0x,f_1x}(\beta(f_0x, f_1x)^+), \\ &\quad 1 - F_{f_0y,f_2y}(\beta(f_0y, f_2y)^+), 1 - F_{f_0x,f_2y}(\beta(f_0x, f_2y)^+), \\ &\quad 1 - F_{f_0y,f_1x}(\beta(f_0y, f_1x)^+)\big\}\big) \\ &\geq f\big(\max\big\{1 - F_{f_0x,f_0y}(t), 1 - F_{f_0x,f_1x}(t), 1 - F_{f_0y,f_2y}(t), 1 - F_{f_0x,f_2y}(t), \\ &\quad 1 - F_{f_0y,f_1x}(t)\big\}\big) \\ &= f\big(1 - \min\big\{F_{f_0x,f_0y}(t), F_{f_0x,f_1x}(t), F_{f_0y,f_2y}(t), F_{f_0x,f_2y}(t), F_{f_0y,f_1x}(t)\big\}\big) \\ &\geq g\big(1 - \min\big\{F_{f_0x,f_0y}(t), F_{f_0x,f_1x}(t), F_{f_0y,f_2y}(t), F_{f_0x,f_2y}(t), F_{f_0y,f_1x}(t)\big\}\big), \end{aligned}$$

a contradiction to (5.5). ■

We can apply the results above in showing the existence of a unique solution of a system of random operator equations.

Let $(\Omega, \mathcal{A}, \mu)$ be a complete probability measure space; and let $(X, |\cdot|)$ be a normed linear space. By $\mathcal{B}$ we mean σ-algebra of Borel subsets of X, so that $(X, \mathcal{B})$ is a measurable space. A mapping $x: \Omega \to X$ is called an X-valued random variable (or generalized random variable), if $x^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$. A mapping $A: \Omega \times X \to X$ is said to be a random operator if for any $x \in X$ $A(., x)$ is a random variable. A random operator A is continuous if for each $\omega \in \Omega$, $A(\omega, .)$ is continuous in the topology induced by the norm $|\cdot|$. The ordered pair $(E, \mathcal{F})$ is an E-space over $(X, |\cdot|)$ if the elements of E are equivalence classes of measurable functions from $(\Omega, \mathcal{A}, \mu)$ into X such that for every $x, y \in E$ and $t \in \mathbb{R}$ the set $\{\omega \in \Omega\colon\ |x(\omega) - y(\omega)| < t\}$ belongs to $\mathcal{A}$, and $\mathcal{F}$ is given via $F_{x,y}(t) := \mu\{\omega \in \Omega\colon\ |x(\omega) - y(\omega)| < t\}$. By [13] it is known that $(E, \mathcal{F}, T_m)$ is a Menger space. In the following we shall assume that $(X, |\cdot|)$ is a Banach space, then $(E, \mathcal{F}, T_m)$ is complete. A random variable $x(\omega) \in E$ is said to be a random fixed point of the random operator $A(\omega, .)$ if $x(\omega) = A(\omega, x(\omega))$, $\forall\ \omega \in \Omega$. If A is continuous, then $A(\omega, x(\omega)) \in E$, whenever $x(\omega) \in E$. Consider the following system of random operator equations

$$\begin{cases} x_0(\omega) = A_0(\omega, x_0(\omega)) + \alpha_0(\omega) \\ x_1(\omega) = A_1(\omega, x_1(\omega)) + \alpha_1(\omega) \\ x_2(\omega) = A_2(\omega, x_2(\omega)) + \alpha_2(\omega) \end{cases} \tag{5.6}$$

where $\alpha_i \in E$, $i = 0, 1, 2$. Let $f_i: \Omega \times X \to X$ be defined by $f_i(\omega, .) := A_i(\omega, .) + \alpha_i(\omega)$, $i = 0, 1, 2$. The corresponding self-mappings of E are defined in a natural way; we shall denote them by the same letters: $(f_i x)(\omega) := A_i(\omega, x(\omega)) + \alpha_i(\omega)$, $i = 0, 1, 2$.

Theorem 5.4. *Let $(\Omega, \mathcal{A}, \mu), (X, |\cdot|), (E, \mathcal{F}, T_m), A_i, \alpha_i, f_i$, $i = 0, 1, 2$ be as above. Assume*

a) $f_1(E) \subset f_0(E)$, $f_2(E) \subset f_0(E)$,
b) f_0, f_1, f_2 *are commuting,*
c) f_0 *is continuous,*
d) *there exists a function $g: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $(g1) - (g3), (\overline{g4})$ such that for all x, y in E and $t > 0$*

$$\begin{aligned} &\mu\{\omega \in \Omega\colon\ |(f_1 x)(\omega) - (f_2 y)(\omega)| \geq g(t)\} \\ &\leq g\big(\max\{\mu\{\omega \in \Omega\colon\ |(f_0 x)(\omega) - (f_0 y)(\omega)| \geq t\}, \\ &\quad \mu\{\omega \in \Omega\colon\ |(f_0 x)(\omega) - (f_1 x)(\omega)| \geq t\}, \\ &\quad \mu\{\omega \in \Omega\colon\ |(f_0 y)(\omega) - (f_2 y)(\omega)| \geq t\}, \\ &\quad \mu\{\omega \in \Omega\colon\ |(f_0 x)(\omega) - (f_2 y)(\omega)| \geq t\}, \\ &\quad \mu\{\omega \in \Omega\colon\ |(f_0 y)(\omega) - (f_1 x)(\omega)| \geq t\}\}\big). \end{aligned} \tag{5.7}$$

Then there exists a unique solution of the system (5.6).

Proof. Obviously (5.7) is equivalent to

$$1 - F_{f_1x,f_2y}(g(t))$$
$$\le g\big(1 - \min\big\{F_{f_0x,f_0y}(t), F_{f_0x,f_1x}(t), F_{f_0y,f_2y}(t), F_{f_0x,f_2y}(t), F_{f_0y,f_1x}(t)\big\}\big),$$

so Theorem 5.3 applies: there exists a unique common fixed point in X for f_0, f_1, f_2, which is a unique solution for (5.6). ∎

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References

1. Tran Thi Lan Anh, On common fixed points for three commuting mappings, *Vietnam J. Math.* **27** (1999) 183–185.
2. Tran Thi Lan Anh, Some common fixed point theorems for mappings in metric and Menger spaces, *Vietnam J. Math.* **28** (2000) 133–142.
3. Tran Thi Lan Anh, Common fixed point theorems for three commuting mappings in metric and Menger spaces, Preprint, Hanoi Institute of Mathematics, 99/47, 1999.
4. Tran Thi Lan Anh and Nguyen Minh Chuong, Generalizations of common fixed point theorems, Preprint, Hanoi Institute of Mathematics, 97/22, 1997.
5. S. S. Chang, A common fixed point theorem for commuting mappings, *Proc. Amer. Math. Soc.* **83** (1981) 645–652.
6. T. H. Chang, Common fixed point theorems in Menger spaces, *Bull. Inst. Math. Acad. Sinica* **22** (1994) 17–29.
7. L. J. Ciric, A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.* **45** (1974) 267–273.
8. K. M. Das and K. V. Naik, Common fixed point theorems for commuting maps on a metric space, *Proc. Amer. Math. Soc.* **77** (1979) 369–373.
9. B. Fisher and K. Iseki, A generalization of a common fixed point theorem, *Math. Japon.* **35** (1990) 1013–1017.
10. B. Fisher and S. Sessa, A fixed point theorem for two commuting mappings, *Nonlinear Functional Analysis and its Applications*: Mathematical and Physical Sciences, 173, Reidel Publishing Company (1986), 223–227.
11. K. Menger, Statistical metrics, *Proc. Nat. Acad. Sci. USA,* **28** (1942) 535–537.
12. B. Schweizer and A. Sklar, Statistical metric spaces, *Pacific J. Math.* **10** (1960) 313–334.
13. H. Sherwood, On E-spaces and their relation to other classes of probabilistic metric spaces, *J. London Math. Soc.* **44** (1969) 441–448.
14. Do Hong Tan, Some common fixed point theorems for mappings of contractive type, *Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak., Ser. Mat.* **25** (1995) 9–22.