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Short Communication

On the Asymptotic Behavior of Solutions of Inhomogeneous Linear Neutral Functional Differential Equations

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This communication is concerned with sufficient conditions for neutral functional differential equations of the form

$$\frac{d}{dt}Dx_t = Lx_t + f(t), \tag{1}$$

where D, L are bounded linear operators from $\mathcal{C} := C([-r, 0], \mathbb{C}^n)$ to $\mathbb{C}^n$, f is an almost (quasi) periodic function in the sense of Bohr, to have almost (quasi) periodic solutions with the same Fourier exponents as the ones of f.

We announce below a result (whose the detailed proof will be given in a separate paper) that if the set of imaginary solutions of the characteristic equations is bounded and the equation has a bounded, uniformly continuous solution, then it has an almost (quasi) periodic solution with the same set of Fourier exponents as the one of the forcing term f. This result extends and complements results in [1, 9, 12, 14 - 16].

This problem has a long history and is one of the main concerns in the qualitative theory of differential equations. In the simplest case of ordinary differential equations, when one deals with the τ-periodicity of the integral $F(t) = \int_0^t f(\xi)d\xi$, where f is τ-periodic and continuous, an easy computations shows that F is τ-periodic if and only if it is bounded. This simple conclusion turns out to be a motivation for numerous works. In this direction, we refer the reader to [1, 9, 12, 15, 17] and the references therein. When f is periodic the method of study in these works is to prove the existence of fixed points of the period maps. However, this method does not work in the more general case with almost periodic f. A new method of study was introduced in [3, 6, 7, 13, 16] to overcome this obstacle which makes use of the so-called evolution semigroups associated with the evolutionary processes generated by equations. In our equation (1), with general assumptions on D and L, the Cauchy problem may have no solutions, so one has no evolutionary processes (or in this case, solution semigroups). In our recent paper [14], we have begun studying conditions for the existence of almost periodic solutions to Eq. (1). We have showed that if (1) has a bounded, uniformly continuous solution, $\Delta_i \backslash Sp(f)$ is closed, and $Sp(f)$ is countable, where $Sp(f)$ is the Beurling spectrum of f, that is equal to the closure of the set of the Fourier exponents of f in the case of almost periodic functions, and

$$\Delta_i := \{\xi \in \mathbb{R} : \det \Delta(i\lambda) = 0\}$$
$$\Delta(\lambda) := \lambda D e^{\lambda \cdot} - L e^{\lambda \cdot}, \quad \lambda \in \mathbb{C},$$

then (1) has an almost periodic solution w such that $Sp(w) \subset Sp(f)$.

We impose the following condition to (1) (*the standing assumption*) $\det \Delta(\lambda) \not\equiv 0$. By using the spectral theory of functions we can show that

Theorem 1. *Under the standing assumption, any bounded and continuous solution of Eq.* (1) *is almost periodic.*

The almost periodicity of bounded solutions of ordinary differential equations has been proved in [2]. For abstract ordinary differential equations in Banach spaces this result has been considered in [8, Chap. 6] with additional conditions.

Next, we estimate the Bohr spectrum of an almost periodic solution to (1).

Theorem 2. *Let x be a bounded and uniformly continuous solution of Eq.* (1). *Then*

$$\sigma_b(f) \subset \sigma_b(x) \subset \Delta_i \cup \sigma_b(f). \tag{2}$$

The main results can be stated as follows:

Theorem 3. *Let Δ_i be bounded. Moreover, assume that Eq.* (1) *has a bounded, uniformly continuous solution. Then, there exists an almost periodic solution w to Eq.* (1) *such that*

$$\sigma_b(w) = \sigma_b(f). \tag{3}$$

A set of reals S is said to have an *integer and finite basis* if there is a finite subset $T \subset S$ such that any element $s \in S$ can be represented in the form $s = n_1 b_1 + \cdots + n_m b_m$, where $n_j \in \mathbf{Z}, j = 1, \cdots, m$, $b_j \in T, j = 1, \cdots, m$. An almost periodic function f is said to be *quasi periodic* if it is of the form $f(t) = F(t, t, ..., t)$, $t \in \mathbb{R}$, where $F(t_1, t_2, ..., t_p)$ is a $\mathbb{C}^n$-valued continuous function of p variables which is periodic in each variable. The function f is quasi periodic if and only if the set of its Fourier-Bohr exponents has an integer and finite basis (see [8, p. 48]).

Corollary 4. *Let all assumptions of the above theorem be satisfied. Moreover, let f be quasi-periodic. If Eq. (1) has a bounded, uniformly continuous solution, then it has a quasi-periodic solution w such that $\sigma_b(w) \subset \sigma_b(f)$.*

Example 5. Consider the equation

$$\dot{x}(t-2) = Bx(t-1) + f(t), \tag{4}$$

where B is an $n \times n$-matrix and f is an almost periodic function. Δ_i is the set of $\lambda \in \mathbb{R}$ such that the matrix $\Delta(i\lambda) = i\lambda I - e^{i\lambda} B$ is not invertible. It is easy to see that for $|\lambda| > \|B\|$, this operator is invertible. Hence, Δ_i is bounded. Applying the above theorem we can conclude that if (4) has a bounded uniformly continuous solution, then it has an almost periodic solution w such that $\sigma_b(w) = \sigma_b(f)$.

Example 6. Consider scalar equations of the form

$$\dot{x}(t) + \sum_{k=1}^{N} A_k \dot{x}(t-\tau_k) = \sum_{j=1}^{M} B_j x(t-\mu_j) + f(t), \quad x(t) \in \mathbb{R} \tag{5}$$

where $N, M \in \mathbb{N}$, A_k, B_j are reals, τ_k, μ_j are positive reals. Then, the characteristic operator is of the form

$$\Delta(\lambda) = \lambda + \lambda \sum_{k=1}^{N} e^{-\tau_k \lambda} A_k - \sum_{j=1}^{M} e^{-\mu_j \lambda} B_j .$$

We now show that the function $\Delta(\lambda) \not\equiv 0$. This is obvious since

$$\lim_{\lambda \in \mathbb{R},\ \lambda \to +\infty} \Delta(\lambda) = +\infty .$$

That is, the standing assumption holds for this class of equations.

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