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Ideals and Idempotents of the Rings of Generalized Power Series

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Abstract. In this paper, we investigate ideals and idempotents of a ring of generalized power series. We show that: $[[(Ra)^{S,\leq}]] = [[R^{S,\leq}]]ca$ and $[[(R/I)^{S,\leq}]] \cong [[R^{S,\leq}]]/[[I^{S,\leq}]]$, and discuss the relation between $[[\sqrt{I}^{S,\leq}]]$ and $\sqrt{[[I^{S,\leq}]]}$. We also characterize the idempotents of a ring of generalized power series.

In [2, 4 - 9], Ribenboim carried out an extensive study of rings of generalized power series. Now we recall the definition of the ring $[[R^{S,\leq}]]$. Let $(S, \leq)$ be an ordered set. $(S, \leq)$ is said to be artinian if every strictly decreasing sequence of elements in S is finite. $(S, \leq)$ is said to be narrow if every subset of pairwise order-incomparable elements of S is finite. A monoid is a commutative semigroup (its operation shall be denoted additively) with a neutral element 0. If S is a monoid, and $\leq$ is a compatible order relation (that is, if $s \leq s'$ and $t \in S$ then $s + t \leq s' + t$), then S is called an ordered monoid. Further, if the order is strict, that is, if $s, s', t \in S$ and $s < s'$ then $s + t < s' + t$, $(S, \leq)$ is called a strictly ordered monoid.

Unless stated otherwise, in this paper $(S, \leq)$ and R will denote a strictly ordered monoid and an associative ring, respectively.

Let $A = [[R^{S,\leq}]]$ be the set of all maps $f : S \to R$ such that $\text{supp}(f) = \{s \in S | f(s) \neq 0\}$ is artinian and narrow as an ordered subset of $(S, \leq)$. With pointwise addition, clearly $\text{supp}(f + g) \subseteq \text{supp}(f) \cup \text{supp}(g)$ and $\text{supp}(-f) =$

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supp(f). It follows that A is an additive group. For any $f, g \in A$ and $s \in S$, the set

$$X_s(f,g) = \{(t,u) \in S \times S | + u = s, \;\; f(t) \neq 0, \;\; g(u) \neq 0\}$$

is finite [7]. This result allows us define the operation of convolution:

$$(fg)(s) = \sum_{(t,u)\in X_s(f,g)} f(t)g(u),$$

from which we can see that $\mathrm{supp}(fg) \subseteq \mathrm{supp}(f) + \mathrm{supp}(g)$.

With this operation, and pointwise addition, A becomes a ring, which is known as the generalized power series ring. The elements of A are called generalized power series with coefficients in R and exponents in S.

For example, if $S = \mathbb{N}$ the addition monoid of integers ≥ 0 under the usual order, then $A \cong R[[x]]$, the usual ring of power series. If S is a group and $\leq$ is the trivial order, then $A = R[S]$, the group-ring of S over R. Further examples are given in [6, 7].

It is easy to see that if R is commutative, then so is A; and if R possesses unit element 1, then A has also. The identity 1_A of A is that:

$$1_A(0) = 1, \quad 1_A(s) = 0 \quad \text{for every} \quad s(\neq 0) \in S.$$

For each $f \in A$, $f \neq 0$, supp(f) is non-empty artinian and narrow. If $(S, \leq)$ is totally ordered, then there exists the smallest element of supp(f), denoted by $\pi(f)$. The following notation we will due to [6]:

Let $r \in R$, define a mapping $c_r \in A$ as follows:

$$c_r(0) = r, \quad c_r(s) = 0 \quad \text{for every} \quad s(\neq 0) \in S.$$

In fact, $c_1 = 1_A$.

1. Ideals

Let R be a ring, S be a strictly ordered monoid, we still denote the generalized power series ring $[[R^{S,\leq}]]$ by A. If I is an ideal (left ideal, or right ideal) of R, let

$$[[I^{S,\leq}]] = \{f \in A \,|\, f(s) \in I \quad \text{for every} \quad s \in S\}.$$

It is easy to verify that $[I^{S,\leq}]]$ is an ideal (left ideal, or right ideal) of A. Of course, not all of ideals of A can be written as $[[I^{S,\leq}]]$ (I is an ideal of R). For example, $A = R[[x]]$, Ax is an ideal of A, which cannot be written as the form $[[I^{S,\leq}]]$.

The following result is obvious.

Proposition 1.1. *Let $I_k (k = 1, ..., n)$ be ideals of R. Then*

$$\bigcap_{k=1}^{n} [[I_k^{S,\leq}]] = \Big[\Big[\Big(\bigcap_{k=1}^{n} I_k\Big)^{S,\leq}\Big]\Big].$$

■

Theorem 1.2. *For any $a \in R, [[(Ra)^{S,\leq}]] = [[R^{S,\leq}]]c a$.*

Proof. Denote $I = Ra$. For any $f \in A$, clearly $fca \in A = [[R^{S,\leq}]]$. Now for every $s \in S$, we have

$$(fca)(s) = f(s)ca(0) = f(s)a \in Ra = I.$$

So $fca \in [[I^{S,\leq}]]$. Hence $Aca \subseteq [[I^{S,\leq}]]$.

Conversely, for any $f \in [[I^{S,\leq}]]$, if $s \in \mathrm{supp}(f)$, then $f(s)(\neq 0) \in I = Ra$, which follows that there exists $r_s \in R$ such that $f(s) = r_s a$.

Let
$$g : S \longrightarrow R :\mapsto \begin{cases} r_s, & s \in \mathrm{supp}(f) \\ 0, & s \in S \setminus \mathrm{supp}(f). \end{cases}$$

Consequently, $\mathrm{supp}(g) = \mathrm{supp}(f)$, so $g \in A$. If $s \in \mathrm{supp}(f)$, $(gca)(s) = g(s)ca(0) = r_s a = f(s)$; If $s \in S \setminus \mathrm{supp}(f), (gca)(s) = 0 = f(s)$. So $f = gca$, that is, $f \in Aca$, hence $[[I^{S,\leq}]] \subseteq Aca$. ■

Theorem 1.3. *Let I be an ideal of R. Then $[[R^{S,\leq}]]/[[I^{S,\leq}]] \cong [[(R/I)^{S,\leq}]]$.*

Proof. Let $\eta : R \to R/I$ be the natural homomorphism. For any $f \in A$, consider the mapping $\eta f : S \to R/I$. It is easy to see that $\mathrm{supp}(\eta f) \subseteq \mathrm{supp}(f)$, so $\mathrm{supp}(\eta f)$ is artinian and narrow. Thus $\eta f \in [[(R/I)^{S,\leq}]]$.

Let $\phi : A \longrightarrow [[(R/I)^{S,\leq}]] : f \mapsto \eta f$.

1) For any $f, g \in A, s \in S$

$$\begin{aligned}(\phi(f+g))(s) &= (\eta(f+g))(s) = (f+g)(s) + I \\ &= (f(s)+I) + (g(s)+I) = (\eta f)(s) + (\eta)(s) \\ &= (\eta f + \eta g)(s) = (\phi(f) + \phi(g))(s).\end{aligned}$$

Thus $\phi(f+g) = \phi(f) + \phi(g)$.

$$\begin{aligned}(\phi(fg))(s) &= (\eta(fg))(s) = \eta((fg)(s)) = \eta\Big(\sum_{(u,v)\in X_s(f,g)} f(u)g(v)\Big) \\ &= \sum_{(u,v)\in X_s(f,g)} \eta(f(u))\eta(g(v)) = \sum_{(u,v)\in X_s(\eta f,\eta g)} (\eta f)(u)(\eta g)(v) \\ &= ((\eta f)(\eta g))(s) = (\phi(f)\phi(g))(s).\end{aligned}$$

Thus $\phi(gf) = \phi(f)\phi(g)$. Hence ϕ is a ring homomorphism.

2) Let $h \in [[(R/I)^{S,\leq}]]$. Then, for every $s \in S$, $h(s) \in R/I$. Since η is the natural homorphism, $\eta^{-1}(h(s)) \neq \emptyset$. Now we take an element r_s from $\eta^{-1}(h(s))$, (if $h(s) = 0$, then let $r_s = 0$).

Let $f : S \longrightarrow R : s \mapsto r_s$. We find that $\mathrm{supp}(f) = \mathrm{supp}(h)$ is artinian and narrow. Thus $f \in A$, and

$$(\phi(f))(s) = (\eta f)(s) = \eta(r_s) = h(s);$$

it follows that $\phi(f) = h$. Hence ϕ is surjective.

3) If $f \in [[I^{S,\leq}]]$, then $f(s) \in I$ for every $s \in S$, so $\eta(f(s)) = 0$, thus $\phi(f) = \eta f = 0$, hence $[[I^{S,\leq}]] \subseteq \mathrm{Ker}\,\phi$.

Let $f \in \mathrm{Ker}\,\phi$. Then $\eta f = \phi(f) = 0$, thus, $\eta(f(s)) = 0$ for every $s \in S$, so $f(s) \in I$ and $f \in [[I^{S,\leq}]]$, hence $\mathrm{Ker}\,\phi \subseteq [[I^{S,\leq}]]$. Therefore, $\mathrm{Ker}\,\phi = [[I^{S,\leq}]]$. Now we have $[[R^{S,\leq}]]/[[I^{S,\leq}]] \cong [[(R/I)^{S,\leq}]]$. ■

In [7], Ribenboim considered a pair of ideals of A : $[[\sqrt{I}^{S,\leq}]]$ and $\sqrt{[[I^{S,\leq}]]}$. Now we investigate their relation.

Let R be a commutative ring and I be an ideal of R. Let $\sqrt{I} = \{a \in R |$ there exists integer $n \geq 1$ such that $a^n \in I\}$. We know that, $\sqrt{I}$ is an ideal of R and contains I. Thus, both $[[\sqrt{I}^{S,\leq}]]$ and $\sqrt{[[I^{S,\leq}]]}$ are ideals of A.

Theorem 1.4. *Let R be a commutative ring and I be an ideal of R, $(S, \leq)$ be a strictly totally ordered monoid. Then* $\sqrt{[[I^{S,\leq}]]} \subseteq [[\sqrt{I}^{S,\leq}]]$.

Proof. Let $f \in \sqrt{[[I^{S,\leq}]]}$. There exists integer $n \geq 1$ such that $f^n \in [[I^{S,\leq}]]$. So $f^n(s) \in I$ for every $s \in S$.

suppose there exists $s \in S$ such that $f(s) \notin \sqrt{I}$. Then for every integer $m \geq 1$, $f(s)^m \notin I$.

Denote $B = \{s \in S \,|\, f(s)^m \notin I$ for every positive integer $m\}$. Clearly, $B \subseteq \text{supp}(f)$, then B is artinian and narrow. Since S is totally ordered, there exists the smallest element in B, which is denoted by s_0.

$$f^n(ns_0) = \sum_{u_1+u_2+\cdots+u_n=ns_0} f(u_1)f(u_2)\ldots f(u_n).$$

Note that the above is a finite sum, we have

$$f^n(ns_0) = f(s_0)^n + \sum_{k=1}^{l} f(u_{k1})f(u_{k2})...f(u_{kn})$$

for every summand $f(u_{k1})f(u_{k2})...f(u_{kn})(k = 1, 2, ..., l)$, we know that $u_{k1} + u_{k2} + \cdots + u_{kn} = ns_0$ and $u_{ki} < s_0$ for some i, so $f(u_{ki}) \in \sqrt{I}$. Now let n_i be a positive integer such that $f(u_{ki})^{n_i} \in I$. Since $f^n(ns_0) \in I$, we have

$$f(ns_0)^{n(n_1+n_2+\cdots+n_l)} = \Big(f^n(ns_0) - \sum_{k=1}^{l} f(u_{k1})f(u_{k2})\cdots f(u_{kn})\Big)^{n_1+n_2+\cdots+n_l} \in I.$$

This contradicts that $s_0 \in B$.

Hence, $f \in [[\sqrt{I}^{S,\leq}]]$.

Notice that the inverse statement is false. ∎

Example 1.5. Let $R_n = \left\{ \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ & \ddots & \ddots & \vdots \\ & & a_1 & a_2 \\ & & & a_1 \end{bmatrix}_{n\times n} |a_i \in \mathbb{Z}, \;\; i = 1, 2, ..., n \right\}$. R_n is a commutative ring with the operation of matrix. Then the direct product $R = \prod_{n=1}^{\infty} R_n$ is also a commutative ring.

Let $S = \mathbb{N}$ with the usual ordered. Then $(S, \leq)$ is a strictly totally ordered monoid. So, $A = [[R^{S,\leq}]] \cong R[[x]]$.

Let $A_n = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ & \ddots & \ddots & \vdots \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}_{n\times n}$. Then $A_n \in R_n$ and ${A_n}^n = 0$.

Let

$$\begin{aligned} B_2 &= 0 \times A_2 \times 0 \times 0 \times \cdots, \\ B_3 &= 0 \times 0 \times A_3 \times 0 \times \cdots, \\ &\cdots \\ B_k &= 0 \times \cdots \times 0 \times A_k \times 0 \times \cdots, \end{aligned}$$

It is easy to show that ${B_k}^k = 0 \in R,\ \ k = 2, 3, \cdots$.

Now let $f = B_2 + B_3 x + B_4 x^2 + \cdots + B_{k+2} x^k + \cdots$, $I = \{0\}$ be the zero ideal of R. Since for any $k \in S$, $f(k) = B_{k+2} \in \sqrt{I}$, so, $f \in [[\sqrt{I}^{S,\leq}]]$. But $f \notin \sqrt{[[I^{S,\leq}]]}$. In fact, for any $n \geq 1$,

$$f^n = {B_2}^n + {B_3}^n x^n + {B_4}^n x^{2n} + \cdots + {B_n}^n x^{(n-2)n} + {B_{n+1}}^n x^{(n-1)n} + \cdots,$$

note that ${B_{n+1}}^n \neq 0$, so $f^n \neq 0$, and then $f \notin \sqrt{[[I^{S,\leq}]]}$. ■

Now, we discuss the case of $[[\sqrt{I}^{S,\leq}]] \subseteq \sqrt{[[I^{S,\leq}]]}$, and give some conditions such that $[[\sqrt{I}^{S,\leq}]] = \sqrt{[[I^{S,\leq}]]}$.

Lemma 1.6. [6] *Let I, I' be ideals of a ring R. Then $[[I^{S,\leq}]] \subseteq [[{I'}^{S,\leq}]]$ if and only if $I \subseteq I'$.*

Proposition 1.7. *Let $(S, \leq)$ be a strictly totally ordered monoid, I be an ideal of a commutative ring R. If $\sqrt{[[I^{S,\leq}]]}$ can be written as $[[J^{S,\leq}]]$ (J is an ideal of R) then $J = \sqrt{I}$ and $[[\sqrt{I}^{S,\leq}]] = \sqrt{[[I^{S,\leq}]]}$.*

Proof. By Theorem 1.4 and Lemma 1.6, $J \subseteq \sqrt{I}$.

On the other hand, let $r \in \sqrt{I}$. Then there exists $n \geq 1$ such that $r^n \in I$, now $c_r^n \in A$:

$$c_r^n(0) = r^n, \quad c_r^n(s) = 0 \quad (s \neq 0).$$

So $c_r^n \in [[I^{S,\leq}]]$ and $c_r \in \sqrt{[[I^{S,\leq}]]} = [[J^{S,\leq}]]$. Thus $r = c_r(0) \in J$, then $\sqrt{I} \subseteq J$. Therefore $J = \sqrt{I}$. ■

Proposition 1.8. *Let I be an ideal of a commutative ring R. If there exists $n \geq 1$ such that $\sqrt{I}^n \subseteq I$, then $[[\sqrt{I}^{S,\leq}]] \subseteq \sqrt{[[I^{S,\leq}]]}$.*

Proof. For any $f \in [[\sqrt{I}^{S,\leq}]]$ and $u \in S$, we know that $f(u) \in \sqrt{I}$. Then for any $s \in S$,

$$f^n(s) = \sum_{u_1+u_2+\cdots+u_n=s} f(u_1)f(u_2)\cdots f(u_n) \in \sqrt{I}^n \subseteq I,$$

consequently, $f^n \in [[I^{S,\leq}]]$, so $f \in \sqrt{[[I^{S,\leq}]]}$ and $[[I^{S,\leq}]] \subseteq \sqrt{[[I^{S,\leq}]]}$. ■

Corollary 1.9. *Let $(S,\leq)$ be a strictly totally ordered monoid, I be an ideal of a commutative ring R. If there exists $n \geq 1$ such that $\sqrt{I}^n \subseteq I$, then $[[\sqrt{I}^{S,\leq}]]ca = \sqrt{[[I^{S,\leq}]]}$.* ■

In particular, if I is a prime ideal, then $\sqrt{I} = I$, thus $[[\sqrt{I}^{S,\leq}]] = \sqrt{[[I^{S,\leq}]]}$.

Lemma 1.10. *Let R be a noetherian commutative ring and I be an ideal of R. Then there exists integer $n \geq 1$ such that $\sqrt{I}^n \subseteq I$.*

Proof. Because R is noetherian, $\sqrt{I}$ is finitely generated. Let $\sqrt{I} = \langle a_1, a_2, ..., a_m\rangle$ with $a_i^{n_i} \in I$, $i = 1, 2, ..., m$. Denote $\sum_{i=1}^{m} n_i = k$.

For any $b_1, b_2, ..., b_k \in \sqrt{I}$, let $b_j = r_{j1}a_1 + r_{j2}a_2 + ... + r_{jm}a_m, \quad j = 1, 2, ..., k$. Then

$$\begin{aligned} b_1 b_2 \cdots b_k &= (r_{11}a_1 + \cdots + r_{1m}a_m)\cdots(r_{k1}a_1 + \cdots r_{km}a_m) \\ &= \sum_{1\leq j_1, j_2, \cdots, j_k \leq m} (r_{1j_1}a_{j_1})(r_{2j_2}a_{j_2})\cdots(r_{kj_k}a_{j_k}) \\ &= \sum_{1\leq j_1, j_2, \cdots, j_k \leq m} (r_{1j_1}r_{2j_2}\cdots r_{kj_k})(a_{j_1}a_{j_2}\cdots a_{j_k}). \end{aligned}$$

Every term of the sum has a form as

$$ra_1^{l_1}a_2^{l_2}\cdots a_m^{l_m}, \quad l_1 + l_2 + \cdots + l_m = k,$$

then there must exist some l_t such that $l_t \geq n_t$, so $a_t^{l_t} \in I$ and $ra_1^{l_1}a_2^{l_2}\cdots a_m^{l_m} \in I$. Now $b_1, b_2 \cdots b_k \in I$. Hence, $\sqrt{I}^k \subseteq I$. ■

Thus, by Lemma 1.10, Proposition 1.8 and Theorem 1.4, we can obtain the following result.

Theorem 1.11. *Let R be a noetherian commutative ring and I be an ideal of $R, (S,\leq)$ be a strictly totally ordered monoid. Then $[[\sqrt{I}^{S,\leq}]] = \sqrt{[[I^{S,\leq}]]}$.* ■

We now consider the order monoid $(S,\leq)$ satisfying the following condition:

$$(S0) \quad 0 \leq s \quad \text{for every} \quad s \in S.$$

Proposition 1.12. *Let $(S,\leq)$ be a strictly totally ordered monoid with $(S0)$, I be an ideal of a commutative ring R. If $f \in [[\sqrt{I}^{S,\leq}]]$, and $T = \{s \in S \mid f(s) \notin I\}$ is finite, then $f \in \sqrt{[[I^{S,\leq}]]}$.*

Proof. Consider T is finite and let $T = \{s_1, s_2, \cdots, s_k\}$. By $f \in [[\sqrt{I}^{S,\leq}]]$, for every s_i $(i = 1, 2, ..., k)$, there exists the smallest positive integer n_i such that

$f(s_i)^{n_i} \in I$. Let $n \in \sum_{i=1}^{k} n_i$; then $f^n \in [[I^{S,\leq}]]$. In fact, for every $s \in S$,

$$f^n(s) = \sum_{u_1+u_2+\cdots+u_n=s} f(u_1)f(u_2)\cdots f(u_n).$$

We can divide the finite summands of $f^n(s)$ into two parts, denoted by $\Sigma_{(1)}$ and $\Sigma_{(2)}$ respectively. For every summand $f(u_1)f(u_2)...f(u_n)$ of $\Sigma_{(1)}, u_1, u_2, \cdots, u_n \in T$, it means that, every u_i is taken from $\{s_1, s_2, \cdots, s_k\}$, then there must be some s_j which is at least taken n_j times, thus $f(u_1)f(u_2)\cdots f(u_n) = f(s_j)^{n_j}Y \in I$. For every summand $f(u_1)f(u_2)\cdots f(u_n)$ of $\Sigma_{(2)}$, there must exist some $u_i \notin T$, then $f(u_i) \in I$, so $f(u_1)f(u_2)\cdots f(u_n) \in I$. Hence $f^n(s) \in I$. Therefore $f \in \sqrt{[[I^{S,\leq}]]}$. ■

Proposition 1.13. *Let $(S, \leq)$ be a strictly totally ordered monoid satisfying $(S0)$, I be an ideal of a commutative ring R. Let $f \in [[\sqrt{I}^{S,\leq}]]$. If $f \notin \sqrt{[[I^{S,\leq}]]}$, then there must exist $g = f^m$ (m is a positive integer) and a sequence $0 < s_1 < s_2 < s_3 \cdots$ in S such that:*

(1) $g^i(s_i) \notin I, \quad i = 1, 2, 3, \cdots$;

(2) *For every $k \geq 1$, there exists $n_k \geq 1$, such that $s_{nk} > ns_k$ for any $n \geq n_k$;*

(3) *For every $n \geq 1$, if $n = k + j$, then $s_n \geq s_k + s_j$, and $s_n = s_k + s_j$ if and only if $g^k(s_k).g^j(s_j) \notin I$.*

Proof. We firstly consider the case of $f(0) \in I$.

Since $f \notin \sqrt{[[I^{S,\leq}]]}$, for every $n \geq 1, f^n \notin [[I^{S,\leq}]]$. Now, let $X_n = \{s \in S \mid f^n(s) \notin I\} \neq \emptyset$. Clearly $X_n (\subseteq \text{supp}(f^n))$ is a artinian and narrow set. By $(S, \leq)$ is totally ordered, we denote the smallest element of X_n by s_n.

Note that $f(0) \in I$, then $s_1 > 0$.

If $n \geq 2$, by $f^n(s_n) \notin I$, $f^n(s_n) = \sum_{u+v=s_n} f^{n-1}(u)f(v)$, and then, there exist $u_1, v_1 \in S, u_1 + v_1 = s_n$ such that $f^{n-1}(u_1)f(v_1) \notin I$. Since $s_n = u_1 + v_1 \geq s_{n-1} + s_1 > s_{n-1}$, we obtain a sequence in S:

$$0 < s_1 < s_2 < s_3 < \cdots$$

(1) By the definition of $s_i, f^i(s_i) \notin I$, $i = 1, 2, ...$

(2) For every $k \geq 1$, since $f \in [[\sqrt{I}^{S,\leq}]]$, $f^k \in [[\sqrt{I}^{S,\leq}]]$. Thus there exists $n_k \geq 1$ such that $(f^k(s_k))^{n_k} \in I$.

For any $n \geq n_k$,

$$f^{nk}(s_{nk}) = \sum_{u_1+\cdots+u_n=s_{nk}} f^k(u_1)f^k(u_2)\cdots f^k(u_n)$$

if for every summand $f^k(u_1)f^k(u_2)\cdots f^k(u_n)$, there is some $u_i < s_k$, then $f^k(u_i) \in I$, it follows that every summand $f^k(u_1)f^k(u_2)\cdots f^k(u_n) \in I$, so $f^{nk}(s_{nk}) \in I$, which is impossible. Now consider

$$f^{nk}(ns_k) = (f^k(s_k))^n + \sum f^k(u_1)f^k(u_2)\cdots f^k(u_n).$$

In $\Sigma f^k(u_1)f^k(u_2)\cdots f^k(u_n)$, every summand satisfies that "$u_1+\cdots+u_n = ns_k$ and there exists some $u_i < s_k$". Thus $\Sigma f^k(u_1)f^k(u_2)\cdots f^k(u_n)\in I$, but $(f^k(s_k))^n \in I$, so $f^{nk}(ns_k)\in I$. Hence $s_{nk} > ns_k$.
(3) If $n = k+j$, then by $f^n(s_n)\notin I$, $f^n(s_n) = \sum\limits_{u+v=s_n} f^k(u)f^j(v)$. We have $s_n \geq s_k + s_j$.

If $s_n = s_k + s_j$, then

$$f^n(s_n) = f^n(s_k+s_j) = f^k(s_k)f^j(s_j) + \sum f^k(u)f^j(v).$$

For every summand of $\Sigma f^k(u)f^j(v)$, u, v satisfy "$u+v = s_k+s_j$ and $u < s_k$ or $v < s_j$". So $\Sigma f^k(u)f^j(v)\in I$. Then by $f^n(s_n)\notin I$, $f^k(s_k)f^j(s_j)\notin I$.

If $f^k(s_k)f^j(s_j)\notin I$. By above, it follows that $f^n(s_k+s_j)\notin I$. So $s_n \leq s_k+s_j$. Since $s_n \geq s_k+s_j$, $s_n = s_k+s_j$.

Secondly we consider the case of $f(0)\notin I$. Since $f(0)\in\sqrt{I}$, there exists $m\geq 1$ such that $f(0)^m\in I$, then $f^m(0) = f(0)^m \in I$. Let $g = f^m$. Then $g(0)\in I$. By $f\in[[\sqrt{I}^{S,\leq}]]$, $g = f^m \in [[\sqrt{I}^{S,\leq}]]$, but $f\notin\sqrt{[[I^{S,\leq}]]}$, so $g = f^m \notin \sqrt{[[I^{S,\leq}]]}$. Now we come back to the first case. ∎

Let $\mathrm{Nil}(R)$ denote the nil radical of R. If R is commutative, $\mathrm{Nil}(R) = \sqrt{0}$, $[[0^{S,\leq}]] = 0$ is the zero-ideal of A. Thus

$$[[\mathrm{Nil}(R)^{S,\leq}]] = [[\sqrt{0}^{S,\leq}]], \quad \mathrm{Nil}\big([[R^{S,\leq}]]\big) = \sqrt{[[0^{S,\leq}]]}.$$

Hence, it follows that all results above adapt to $[[\mathrm{Nil}(R)^{S,\leq}]]$ and $\mathrm{Nil}\big([[R^{S,\leq}]]\big)$. If "$\leq$" is a total order on S, $[[\mathrm{Nil}(R)^{S,\leq}]] \supseteq \mathrm{Nil}\big([[R^{S,\leq}]]\big)$. But the inverse is not true (see Example 1.5).

We recall that the nil ideal of a commutative artinian ring is nilpotent. By Corollary 1.9, we have

Corollary 1.14. *If $(S,\leq)$ is a strictly totally ordered monoid, R is a commutative artinian ring, then* $[[\mathrm{Nil}(R)^{S,\leq}]] = \mathrm{Nil}\big([[R^{S,\leq}]]\big)$. ∎

2. Idempotents

In this section, we pay attention to the idempotents of $A = [[R^{S,\leq}]]$. In the following, we let R be a ring with identity 1, thus A possesses identity, namely 1_A.

Firstly, we give some notations: $\mathrm{Id}(R) = \{$all idempotents of $R\}$, $\mathrm{Cen}(R) = \{r\in R \mid rx = xr$ for every $x\in R\}$, $U(R) = \{$all units of $R\}$. A ring R is called normal if $\mathrm{Id}(R)\subseteq \mathrm{Cen}(R)$. Now, we have

Theorem 2.1. *Let $(S,\leq)$ be strictly totally ordered and satisfy condition (S0), R be a normal ring. If $f\in A$, then f is an idempotent of A if and only if there be an idempotent e in R such that $f = c_e$.*

Proof. ($\Leftarrow$) We shall proof that if $e\in\mathrm{Id}(R)$, then $c_e\in\mathrm{Id}(A)$.

In fact, if $e \in \mathrm{Id}(R)$, then

$$(c_e c_e)(0) = c_e(0)c_e(0) = ee = e = c_e(0).$$

For any non-zero $s \in S$, $(c_e c_e)(s)$=0=$c_e(s)$. So $c_e c_e$=c_e, and then $c_e \in \mathrm{Id}(A)$.

($\Rightarrow$) If $f = 0 \in A$, then let $e = 0 =\in R$. So, $f = c_e$.

If $f \in A$ and $f \neq 0$. Since f is an idempotent, $f = f^2$. Then $f(0) = f^2(0) = f(0).f(0)$.

Let $e = f(0)$. Then $e \in \mathrm{Id}(R)$. Now we have to proof $f = c_e$.

By $f \neq 0$, $f(0) \neq 0$. In fact, if $f(0) = 0$, then $s_0 = \pi(f) > 0$. It is clear to see that $X_{s_0}(f, f) = \emptyset$, so $f(s_0) = f^2(s_0) = 0$. This is impossible.

For every non-zero $s \in S$, by $f = f^2$, we know $f(s) = f^2(s)$. Consider $X_s(f, f) = \{(u, v) \,|\, u+v = s, u, v \in \mathrm{supp}(f)\}$. Since S is strictly totally ordered, any (u, v) in $X_s(f, f)$ satisfies $0 \leq u, v \leq s$. If there exists u and $0 < u < s$, then note that $M = \{u \,|\, 0 < u, v < s, (u, v) \in X_s(f, f)\}$, $M \neq \emptyset$. Since $X_s(f, f)$ is finite, M is a finite set. Thus M has the smallest element, namely u_0. Clearly, $X_{u_0}(f, f) = \{(0, u_0), (u_0, 0)\}$. Then, by $f \in A, f(0) \in \mathrm{Cen}(R)$,

$$f(u_0) = f^2(u_0) = f(0)f(u_0) + f(u_0)f(0) = 2f(0)f(u_0)$$
$$f(0)f(u_0) = 2(f(0))^2 f(u_0) = 2f(0)f(u_0).$$

So $f(0)f(u_0) = 0$, and then $f(u_0) = 0$, this contradicts the assumption that $u_0 \in \mathrm{supp}(f)$. Now consider $f(s)$. If $f(s) \neq 0$, then $X_s(f, f) = \{(0, s), (s, 0)\}$, and

$$f(s) = f^2(s) = f(0)f(s) + f(s)f(0) = 2f(0)f(s),$$
$$f(0)f(s) = 2(f(0))^2 f(s) = 2f(0)f(s).$$

So $f(0)f(s) = 0$, and so $f(s) = 0$, which is contradiction.

Therefore, $f(0) \neq 0$ and $f(s) = 0$ for any non-zero $s \in S$. Hence $f = c_e$. ∎

Corollary 2.2. *Let $(S, \leq)$ be strictly totally ordered and satisfy condition $(S0)$. Then A is normal if and only if R is normal.*

Proof. ($\Leftarrow$) By Theorem 2.1, for any $f \in \mathrm{Id}(A)$, there exists $e \in \mathrm{Id}(R)$ such that $f = c_e$. Note that $\mathrm{Id}(R) \subseteq \mathrm{Cen}(R)$, we have $e \in \mathrm{Cen}(R)$.

For any $g \in A$, $s \in S$,

$$(fg)(s) = (c_e g)(s) = c_e(0)g(s) = eg(s) = g(s)e = (gc_e)(s) = (gf)(s)$$

So $fg = gf$. Hence $f \in \mathrm{Cen}(A)$. Hence $\mathrm{Id}(A) \subseteq \mathrm{Cen}(A)$, and so A is *normal*.

($\Rightarrow$) For any $e \in \mathrm{Id}(R)$, $(c_e c_e)(0) = c_e(0)c_e(0) = e^2 = e = c_e(0)$, and for every $s(\neq 0) \in S$, $(c_e c_e)(s) = 0$. So c_e is an idempotent(in A) and $c_e \in \mathrm{Cen}(A)$. Thus, for any $r \in R$, we have $c_r c_e = c_e c_r$, and so $re = er$ and $e \in \mathrm{Cen}(R)$.
Hence R is normal.

Recall that two idempotents e_1, e_2 of a ring R are called orthogonal, if $e_1e_2 = 0 = e_2e_1$. A non-zero idempotent $e \in R$ is called primitive, if e can not be written as a sum of two non-zero orthogonal idempotents.

Lemma 2.3. [10, p.143, Theorem 5] *Let $e \in R$ be an idempotent. Then e is primitive if and only if for any idempotent $r \in R$, $r = e$ whenever $r = er = re$.* ∎

Corollary 2.4. *Let $(S, \leq)$ be strictly totally ordered and satisfy condition $(S0)$, R be a normal ring, $f \in A$. Then $f \in A$ is a primitive idempotent if and only if there exists a primitive idempotent $e \in R$ such that $f = c_e$.*

Proof. $(\Rightarrow)$ suppose that f is a primitive idempotent of A. Then, by Theorem 2.1, there exists an idempotent $e \in R$ such that $f = c_e$.

Suppose idempotent $r \in R$ satisfies $r = er = re$. Hence $c_r \in \mathrm{Id}(A)$, and $c_r(0) = r = er = c_e(0)c_r(0) = (c_ec_r)(0)$. Similarly, $c_r(0) = (c_ec_r)(0)$. For any $s \in S, s \neq 0$, $c_r(s) = 0$, $(c_ec_r)(s) = 0 = (c_rc_e)(0)$, $c_r(s) = (c_ec_r)(s) = (c_rc_e)(s)$. Thus $c_r = c_ec_r = c_rc_e$. Since $c_e = f$ is a primitive idempotent, by Lemma 2.3, $c_r = c_e$, then $r = e$, it follows that r is a primitive idempotent.

$(\Leftarrow)$ suppose $f = c_e, e \in R$ is a primitive idempotent. For any idempotent $c_r \in A(r \in \mathrm{Id}(R))$, if $c_r = fc_r = c_rf$, then $r = er = re$, so $r = e$, and so $c_r = c_e = f$. It follows that $f \in A$ is a primitive idempotent. ∎

A ring R is called to be local, if R has a unique maximal left ideal. A ring R is local if and only if for any $x \in R$, either x or $1 - x$ is invertible. Other equivalent conditions of a local ring can be found in [1, p.170, Prop. 15.15]. Let $e \in R$ be an idempotent, e is called local if eRe is a local ring.

P. Ribenboim gave a useful result of units of a generalized power series ring:

Lemma 2.5. [7] *Assume that S satisfies condition $(S0)$ and let $f \in A$. Then $f \in U(A)$ if and only if $f(0) \in U(R)$.*

By this result, we obtain:

Proposition 2.6. *If S satisfies condition $(S0)$, then A is a local ring if and only if R is local.*

Proof. $(\Rightarrow)$ For any $a \in R$, since A is local, either c_a or $1_A - c_a$ is a unit of A. Note that $c_a(0) = a$, $(1_A - c_a)(0) = 1 - a$, by Lemma 2.5, either a or $1 - a$ is a unit of R. So R is local.
$(\Leftarrow)$ For any $f \in A$, clearly $f(0) \in R$. Since R is a local ring, so either $f(0)$ or $1 - f(0)$ is invertible, then by Lemma 2.5, either f or $1_A - f$ is invertible, so A is local. ∎

Corollary 2.7. *Let $(S, \leq)$ be strictly totally ordered and satisfy condition $(S0)$, R be a normal ring, $f \in A$. Then $f \in A$ is a local idempotent if and only if there exists a local idempotent $e \in R$ such that $f = c_e$.*

Proof. (⇒) Assume that $e \in R$ is a local idempotent, then eRe is a local ring. By $e \in \mathrm{Id}(R) \subseteq \mathrm{Cen}(R)$, $eRe = Re$, so Re is a local ring. By Proposition 2.6, $[[Re^{S,\leq}]]$ is a local ring. By Theorem 1.2, $[[Re^{S,\leq}]] = [[R^{S,\leq}]]c_e = Ac_e$, thus Ac_e is local. By Theorem 2.1 and Corollary 2.2, $c_e \in \mathrm{Id}(A) \subseteq \mathrm{Cen}(A)$, then $c_eAc_e = Ac_e$ is a local ring, hence $f = c_e$ is a local idempotent of A.
(⇐) If $f \in A$ is a local idempotent, by Theorem 2.1, there exists an $e \in \mathrm{Id}(R)$ such that $f = c_e$. Thus $c_e \in A$ is a local idempotent, then $Ac_e = c_eAc_e$ is a local ring. And then, $[[Re^{S,\leq}]] = Ac_e$ is a local ring, so Re is a local ring, and eRe is local. Hence e is a local idempotent of R. ∎

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