Vietnam Journal of Mathematics **31**:3 (2003) 333-340

Vietnam Journal
of
MATHEMATICS
© NCST 2003

Separately Holomorphic Functions with Pluripolar Singularities

Omar Alehyane[1] **and Hichame Amal**[2]

[1] *Université Chouaib Doukkali, Département de Mathématique, Faculté des science, B. P. 20, El Jadida, Maroc*
[2] *Université Mohammed V, Département de Mathématiques, Faculté des Sciences, B. P 1014, Rabat, Maroc*

Received July 24, 2002

Abstract. In this paper, we show that if f is a separately holomorphic function on $\mathcal{X} \setminus P$, where $\mathcal{X} := E \times V \cup U \times F$ with $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ are domains, $E \subset U$ and $F \subset V$ are locally pluriregular set, and P is a closed pluripolar set in an open neighborhood W of $\mathcal{X}$, then there is an open neighborhood Ω of $\mathcal{X}$, a closed pluripolar set S in Ω and a function $\widehat{f} \in \mathcal{O}(\Omega \setminus S)$ such that $S \cap \mathcal{X}^* \subset P$ and $\widehat{f}_{|\mathcal{X}^* \setminus P} = f_{|\mathcal{X}^* \setminus P}$ for some subset $\mathcal{X}^*$ of $\mathcal{X}$ so that $\mathcal{X} \setminus \mathcal{X}^*$ is pluripolar.

1. Introduction

Let $U \subset \mathbb{C}^n$ be a domain and $E \subset U$ a non-empty set. We let

$$u_{E,U}(x) = \sup\{u(x) \,/\, u \in Psh(U),\, u \le 0 \quad \text{on} \quad E,\;\; u \le 1 \quad \text{on} \quad U\},$$

where $Psh(U)$ denotes the set of plurisubharmonic functions on U, and let $w(., E, U)$ be the relative extremal function of E related to U, as the upper semi-continuous regularization of $u_{E,U}$, i.e $w(., E, U) = u^*_{E,U}$. Note that $w(., E, U)$ is plurisubharmonic on U.

Put

$$\widetilde{w}(., E, U) := \lim_{j \to +\infty} w(., E \cap U_j, U_j),$$

where $(U_j)_{j \ge 1}$ is an exhaustive sequence of relatively compact open set in U. This limit is independent of the choice of the sequence $(U_j)_{j \ge 1}$.

We say that E is locally pluriregular at a point $a \in \overline{E} \cap U$ if and only if $w(a, E \cap D, D) = 0$ for all $D \in \mathcal{B}(a)$, where $\mathcal{B}(a)$ is an arbitrary neighborhood basis of a. Also we say that E is locally pluriregular if E is locally pluriregular at every point $a \in E$.

Let $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^m$ be two domains and $E \subset U$, $F \subset V$ be two non-empty subsets. We define a cross of $\mathbb{C}^{n+m}$ associated to (U, V, E, F) by $\mathcal{X} := U \times F \cup E \times V$. Then one can remark that $\mathcal{X}$ is connected. For a cross $\mathcal{X}$ we put

$$\widehat{\mathcal{X}} := \{(z, w) \in U \times V \,/\, \widetilde{w}(z, E, U) + \widetilde{w}(w, F, V) < 1\}.$$

If U and V are pseudoconvex, then so is $\widehat{\mathcal{X}}$. Moreover, if E and F are locally pluriregular then $\widehat{\mathcal{X}}$ is connected (see [5]).

Let P be a closed pluripolar subset of an open neighborhood W of $\mathcal{X}$. We say that a function $f : \mathcal{X} \setminus P \to \mathbb{C}$ is separately holomorphic if:
1. $f_z = f(z, .) \in \mathcal{O}(V \setminus P_z)$ for all $z \in E$,
2. $f^w = f(., w) \in \mathcal{O}(U \setminus P^w)$ for all $w \in F$,
where $P_z = \{w \in V \,/\, (z, w) \in P\}$ and $P^w = \{z \in U \,/\, (z, w) \in P\}$. When this holds, we write $f \in \mathcal{O}_s(\mathcal{X} \setminus P)$.

Let $u \in Psh(U \times V)$ with $u \not\equiv -\infty$ such that $P \subset \{(z, w) \in U \times V \,/\, u(z, w) = -\infty\}$.

For $(\alpha, \beta) \in U \times V$ with $u(\alpha, \beta) \neq -\infty$, we denote $A_\alpha := \{w \in V \,/\, u(\alpha, w) = -\infty\}$ and $A_\beta := \{z \in U \,/\, u(z, \beta) = -\infty\}$. Obviously for every point $z \in U \setminus A_\alpha$ (resp. $w \in V \setminus A_\beta$) P_z (resp. P^w) is pluripolar.

Let $E^* = \{z \in E \,/\, P_z \text{ is pluripolar}\}$ and $F^* = \{w \in F \,/\, P^w \text{ is pluripolar}\}$. Then $E \setminus E^*$ and $F \setminus F^*$ are pluripolar. Put $\mathcal{X}^* = E^* \times V \cup U \times F^*$. It is clear that $\mathcal{X} \setminus \mathcal{X}^*$ is pluripolar.

The main result of this paper is the following:

Theorem 1.1. *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be two domains $E \subset U$ and $F \subset V$ be locally pluriregular sets, and P a closed pluripolar subset of an open neighborhood W of $\mathcal{X} := E \times V \cup U \times F$. Then for every function $f \in \mathcal{O}_s(\mathcal{X} \setminus P)$, there exist an open neighborhood Ω of $\mathcal{X}$, a closed pluripolar set S of Ω and $\widehat{f} \in \mathcal{O}(\Omega \setminus S)$ such that $S \cap \mathcal{X}^* \subset P$ and a function $\widehat{f}_{|\mathcal{X}^* \setminus P} = f_{|\mathcal{X}^* \setminus P}$.*

The case $P = \emptyset$ was studied by many authors: Siciak [14, 15], Zahariuta [17], Shiffman [13], Nguyen-Zeriahi [11], Nguyen [10] and Alehyane-Zeriahi [1]. In what follows, we need the following results:

Theorem 1.2. (see [1]) *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be two domains $E \subset U$ and $F \subset V$ be locally pluriregular sets. Put $\mathcal{X} = E \times V \cup U \times F$. Then for every function $f \in \mathcal{O}_s(\mathcal{X})$, there exist an open neighborhood Ω of $\mathcal{X}$ and a function $\widehat{f} \in \mathcal{O}(\Omega)$ such that $\widehat{f}_{|\mathcal{X}} = f$.*

The case where $P \neq \emptyset$ was studied by Oktem [8, 9], Siciak [16], Plfug-Jarnicki [5, 6]. In a recent preprint [7], Pflug and Jarnicki have studied the pluripolar case with different techniques. Our approach, inspired by [3], is based on the following property of pseudoconcave sets proved by Sadullaev.

Recall that a set S of $\mathbb{C}^m$ is said to be pseudoconcave if for each point $z \in S$ there is an open neighborhood V of z such that $V \setminus S$ is pseudoconvex. Let Δ^m be the polydisk of $\mathbb{C}^m$, $\Delta^{m-1} = \Pi_{i=1}^{m-1} \Delta_i$ and $\Delta = \Delta_i$ the unit disc of $\mathbb{C}$.

Theorem 1.3. (see [12]) *Let $S \subset \Delta^m$ be a pseudoconcave set such that $\bar{S} \cap (\Delta^{m-1} \times \partial\Delta) = \emptyset$ and $E \subset \Delta^{m-1}$ a non pluripolar set. If $S_z = \{w \in \Delta \,/\, (z,w) \in S\}$ is polar for all $z \in E$, then S is pluripolar.*

Theorem 1.4. (see [2]) *Let Ω be a domain of $\mathbb{C}^n$, $\widehat{\Omega}$ its envelope of holomorphy and S a closed pluripolar set of Ω. Then there exists a closed pluripolar set $\widehat{S}$ of $\widehat{\Omega}$ such that $\widehat{S} \cap \Omega \subset S$ and $\widehat{\Omega} \setminus \widehat{S}$ is the envelope of holomorphy of $\Omega \setminus S$.*

Note that in the analytic case, the problem is completely resolved (see [6]), but in the merely pluripolar one, the problem remains open.

2. Proof of the Main Theorem

We need first to establish some lemmas:

Lemma 2.1. *The set $\mathcal{X}^* \setminus P$ is archwise connected and locally pluriregular.*

Proof. Let $(a,b),(c,d) \in \mathcal{X}^* \setminus P$. Then we have two cases to examine:
i) If $(a,b),(c,d) \in E^* \times V \setminus P$, then $b \in V \setminus P_a$, hence (a,b) can be joined inside $V \setminus P_a$ by (a,b'), with $b' \in V \setminus (P_a \cup P_c)$. As $a,c \in U \setminus P^{b'}$, (a,b') can be joined inside $U \setminus P^{b'}$ by (c,b'). Also $d,b' \in V \setminus P_c$, thus (c,b') can be joined inside $V \setminus P_c$ by (c,d). This implies that inside $\mathcal{X}^* \setminus P$ we can join (a,b) by (c,d).
ii) If $(a,b) \in E^* \times V \setminus P$ and $(c,d) \in U \times F^* \setminus P$ then there are ε and η such that $\Delta^n(c,\varepsilon) \times \Delta^m(d,\eta) \subset W \setminus P$, hence we can find $b' \in \Delta^m(d,\eta) \cap F^*$ to have $a,c \in U \setminus P^{b'}$. As $b' \in V \setminus P_a$, (a,b) can be joined inside $V \setminus P_a$ by (a,b'). Also (a,b') can be joined inside $U \setminus P^{b'}$ by (c,b'), hence from (i), (c,b') can by joined inside $\mathcal{X}^* \setminus P$ by (c,d).

By the fact that E^* and F^* are locally pluriregular, and by Corollary 6 of [4], $\mathcal{X}^*$ is locally pluriregular. Since P is pluripolar, the result follows. ■

Lemma 2.2. *There exists an open neighborhood G of $\mathcal{X}^* \setminus P$ and a function $\widetilde{f} \in \mathcal{O}(G)$ such that $\widehat{f}_{|\mathcal{X}^* \setminus P} = f_{|\mathcal{X}^* \setminus P}$.*

Proof. Let $(a,b) \in (E^* \times V) \setminus P$. Since P_a is a closed and pluripolar set of V, there exists a relatively compact domain O in $V \setminus P_a$ such that $b \in O$ and $F^b = F^* \cap O$ is non-empty. Since $\{a\} \times \bar{O} \subset W \setminus P$, we can find an open connected neighborhood D of the point a such that $D \times O \subset W \setminus P$.

Let $E^a = E \cap D$ and $\mathcal{X}_{ab} = D \times F^b \cup E^a \times O$. Since E^a and F^b are locally pluriregular and $f_{|\mathcal{X}_{ab}} \in \mathcal{O}_s(\mathcal{X}_{ab})$, by Theorem 1.2 there exist an open neighborhood Ω_{ab} of $\mathcal{X}_{ab}$ and $\widetilde{f}_{ab} \in \mathcal{O}(\Omega_{ab})$ such that $\widetilde{f}_{ab|\mathcal{X}_{ab}} = f_{|\mathcal{X}_{ab}}$. Let $\mathbb{B}_{ab}$ be a ball in Ω_{ab} centered at (a,b). Put $G_1 = \bigcup_{(a,b)\in(E^*\times V)\setminus P} \mathbb{B}_{ab}$. In the same way we construct $G_2 = \bigcup_{(a,b)\in(U\times F^*)\setminus P} \mathbb{B}_{ab}$. The needed neighborhood G is the union of G_1 and G_2.

Let $\zeta = (a, b)$ and $\eta = (c, d)$ in $\mathcal{X}^* \setminus P$. We must prove that $\widetilde{f}_{ab} = \widetilde{f}_{cd}$ in $\mathbb{B}_{ab} \cap \mathbb{B}_{cd}$.

From Lemma 2.1, (a, b) can be joined with (c, d) inside $\mathcal{X}^* \setminus P$ by a piecewise differentiable path γ.

Let $(x_1, y_1), ..., (x_k, y_k)$ be in γ such that:

1) $(a, b) = (x_1, y_1)$ and $(x_k, y_k) = (c, d)$.
2) $(x_i, y_i) \in \mathbb{B}_{x_{i+1}y_{i+1}}$, for $i \in \{1, ..., k-1\}$.

Obviously the set $A_i = \mathbb{B}_{x_iy_i} \cap \mathbb{B}_{x_{i+1}y_{i+1}} \cap \mathcal{X}^* \setminus P$ in not empty. Moreover, $\mathcal{X}^* \setminus P$ is locally pluriregular, so A_i is not pluripolar. Since $\mathbb{B}_{x_iy_i} \cap \mathbb{B}_{x_{i+1}y_{i+1}}$ is connected, then there is a function $g \in \mathcal{O}(\bigcup_{i=1}^k \mathbb{B}_{x_iy_i})$ such that $g_{\mathbb{B}_{x_iy_i}} = \widetilde{f}_{x_iy_i|\mathbb{B}_{x_iy_i}}$. It follows that $\widetilde{f}_{ab} = \widetilde{f}_{cd}$ in $\mathbb{B}_{ab} \cap \mathbb{B}_{cd}$. ■

Let $\widehat{G}$ be the envelope of holomorphy of $\widetilde{G}$ and f the holomorphic extension of $\widetilde{f}$ to $\widehat{G}$. For every pseudoconvex domain D, the open set $D' = \widehat{G} \cap D$ is also pseudoconvex and then $S = D \setminus D'$ is pseudoconcave. ■

Let $b \in V$ and assume that for every $z \in U$ there are an open neighborhood O_z of z, $\eta > 0$ and a pluripolar set S_z such that $O_z \times \Delta(b, \eta) \setminus S_z \subset \widehat{G}$. Let $\chi_b : \mathbb{C}^m \to \mathbb{C}^m$, $\chi_b(w) = w - b$. Put $\widehat{G}_b := \chi_b(\widehat{G})$. For $z \in U$ we denote by $\mathcal{R}_z$ the set of positive numbers r for which there exist an open neighborhood O_z of z and a pluripolar set S_z such that $(O_z \times \Delta(r)) \setminus S_z \subset \widehat{G}_b$. Let $R(z) := \sup\{r > 0 \,/\, r \in \mathcal{R}_z\}$. Note that $R_{f,b}(z) = \liminf_{\zeta \to z} R(\zeta)$ is lower semi-continuous and $R_{f,b}(z) > 0$ for all $z \in U$.

We write $M_{f,b} = \{(z, w) \in U \times V \,/\, |w| < R_{f,b}(z)\}$.

Lemma 2.3. *The function $-\log R_{f,b}$ is plurisubharmonic in U.*

Proof. The proof is essentially the same as given by Oktem in [9, Lemma 4.2]. We begin by proving the case $n = m = 1$.

We must prove that for any closed disk $B \subset U$, and for any harmonic function h in an open neighborhood of $\bar{B}$,

$$-\log R_{f,b}(z) \le h(z) \quad \text{for all} \quad z \in \partial B$$
$$\text{implies that} \quad -\log \quad R_{f,b}(z) \le h(z) \quad \text{for all} \quad z \in B.$$

We can assume that $B = \bar{\Delta}$ (the unit disc of $\mathbb{C}$). Let D be an open neighborhood of $\bar{\Delta}$ and $h \in \mathcal{H}(D)$. There exists $g \in \mathcal{O}(D)$ such that $h = Re(g)$. We define $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ by $\phi(z, \zeta) = (z, w)$, where $w = \zeta e^{-g(z)}$, ϕ is biholomorphic.

Our object is to find an open neighborhood H of $\phi(\bar{\Delta} \times \Delta)$ and a pluripolar set Z such that $H \setminus Z \subset \widehat{G}_b$.

If $(z, \zeta) \in \partial\Delta \times \Delta$ and $w = \zeta e^{-g(z)}$, then

$$|w| = |\zeta e^{-g(z)}| = |\zeta| e^{-Reg(z)} = |\zeta| e^{-h(z)} < e^{-h(z)} \le R_{f,b}(z),$$

so $\phi(\partial\Delta \times \Delta) \subset M_{f,b}$. It follows that there is an open neighborhood H' of $\phi(\partial\Delta \times \Delta)$ and a pluripolar set Z' such that $H' \setminus Z' \subset \widehat{G}_b$. Moreover $\phi(z, 0) = (z, 0)$ so $\phi(U \times \{0\}) = U \times \{0\}$.

As $R_{f,b}(z) > 0$ for every $z \in U$ and $\bar{\Delta} \subset\subset U$, there exist $\varepsilon > 0$ and a pluripolar set S_ε such that $\phi(\bar{\Delta} \times \Delta(0,\varepsilon)) \setminus S_\varepsilon \subset \widehat{G}_b$.

Let O be the open set of points $w \in \Delta$ for which there are an open neighborhood O^w of w and a pluripolar set S^w such that $\phi(\bar{\Delta} \times O^w) \setminus S^w \subset \widehat{G}_b$. It follows that we can construct an open neighborhood H'' of $\phi(\bar{\Delta} \times O)$ and a pluripolar set Z'' such that $H'' \setminus Z'' \subset \widehat{G}_b$. Let $H''' = H' \cup H''$ and $Z''' = Z' \cup Z''$, then $H''' \setminus Z''' \subset \widehat{G}_b$. After shrinking, Z''' must be supposed closed in $\widehat{G}_b$.

Our purpose is to prove that $\Delta \cap \partial O$ is polar; this implies that there is an open neighborhood H of $\phi(\bar{\Delta} \times \Delta)$ and a pluripolar set Z such that $H \setminus Z \subset \widehat{G}_b$. And as $\phi(\bar{\Delta} \times \Delta) = \{(z,w) \in \bar{\Delta} \times \mathbb{C} : |w| < e^{-h(z)}\}$, then by definition of $R_{f,b}$ we have $e^{-h(z)} \leq R_{f,b}(z)$ for all $z \in \Delta$, this implies:

$$-\log R_{f,b}(z) \leq h(z) \quad \text{for all} \quad \in \Delta.$$

Assume that $\Delta \cap \partial O$ is not polar and let $K \subset \Delta \cap \partial O$ be a compact and non-polar set. From above there are an open neighborhood D_1 of $\partial\Delta$ and D_2 of K such that $\phi(D_1 \times D_2) \subset H'''$. Put $Z_1 = \phi^{-1}(Z''') \cap (D_1 \times D_2)$ and let $v \in Psh(D_1 \times D_2)$ such that $Z_1 \subset \{x \in D_1 \times D_2 \,/\, v(x) = -\infty\}$ with $v \not\equiv -\infty$. Take $(\alpha,\beta) \in D_1 \times D_2$ with $v(\alpha,\beta) \neq -\infty$ and put $A_\alpha = \{w \in D_2 \,/\, v(\alpha,w) = -\infty\}$. Since A_α is polar, there is $c \in K$ such that $c \notin A_\alpha$. It follows that the set ${Z_1}^c = \{z \in D_1 \,/\, (z,c) \in Z_1\}$ is polar and closed, hence for some $\rho > 1$ the circle $S(0,\rho)$ does not intersect ${Z_1}^c$ and $S(0,\rho) \subset D_1$.

Let $M \subset D_1$ be an open neighborhood of $S(0,\rho)$ and $\eta > 0$ such that $\phi(M \times \Delta(c,\eta)) \subset \widehat{G}_b$; we have $M \times \Delta(c,\eta) \cap Z_1 = \emptyset$.

Put $\Sigma := \phi(\Delta(0,\rho) \times \Delta(c,\eta)) \cap \widehat{G}_b$ and $\Sigma' = \phi^{-1}(\Sigma)$. Obviously the set $Z_2 = (\Delta(0,\rho) \times \Delta(c,\eta)) \setminus \Sigma'$ is pseudoconcave and $\bar{Z}_2 \cap (\partial\Delta(0,\rho) \times \Delta(c,\eta)) = \emptyset$.

Moreover, $\phi([D_1 \cup \bar{\Delta}] \times [\Delta(c,\eta) \cap O]) \subset H'''$ so there is a non-pluripolar set $L \subset \Delta(c,\eta) \cap O$ such that the fiber $Z_1(w) = \{z \in D_1 \cup \bar{\Delta} \,/\, (z,w) \in Z_1\}$ is polar for every $w \in L$. Since ${Z_2}^w \subset Z_1(w)$ for all $w \in L$, Z_2 is pluripolar by Theorem 1.3. It then follows that $c \in O$, and this is a contradiction.

The case $n, m \geq 1$ is treated in the same way as in [9]. ∎

Lemma 2.4. *Let $D \subset \mathbb{C}^n$ be a domain and $A \subset D$ be a locally pluriregular set. Let $A^* \subset A$ with $A \setminus A^*$ pluripolar and φ a plurisubharmonic function in D. If $\varphi \leq 0$ on A^*, then $\varphi \leq 0$ on A.*

Proof. Let $a \in A \setminus A^*$; assume that $\varphi(a) = \lambda > 0$. Let $v = \varphi/2\lambda$ and $\Omega_\lambda := \{z \in D\,;\, v(z) < 1\}$. We have $A^* \cup \{a\} \subset \Omega_\lambda$, $v \leq 0$ in A^* and $v \leq 1$ in Ω_λ. Then $v \leq w_{A^*,\Omega_\lambda}$, so $w_{A^*,\Omega_\lambda}(a) > 0$. Since $(A \cap \Omega_\lambda) \setminus A^*$ is pluripolar then $w_{A\cap\Omega_\lambda,\Omega_\lambda}(a) > 0$; this is impossible since A is locally pluriregular. ∎

Proof of the theorem. By induction on m we are going to prove the existence of an open neighborhood Ω_1 of $E \times V$ and a pluripolar set S_1 such that $\Omega \setminus S_1 \subset \widehat{G}$.

Assume $m = 1$:

(i) Let $a \in E^*$, $b \in V$ and $r > 0$ such that $\Delta(b,r) \subset V$. Since P_a is closed and polar, then there is $\rho \in]0,r[$ such that the circle $S(b,\rho)$ does not intersect P_a. Let $\varepsilon > 0$ and D' be an open neighborhood $S(b,\rho)$ such that $\Delta^n(a,\varepsilon) \times D' \subset \widehat{G}$. Obviously $S_{a,b} = \Delta^n(a,\varepsilon) \times \Delta(b,\rho) \setminus H$ is pseudoconcave, with $H = \Delta^n(a,\varepsilon) \times$

$\Delta(b,\rho)\cap\widehat{G}$, also $S_{a,b}(z)=\{w\in\Delta(b,\rho)\,/\,(z,w)\in S_{a,b}\}\subset P_z$ for every $z\in E^*\cap\Delta^n(a,\varepsilon)$ so $S_{a,b}(z)$ is polar and since $\bar{S}_{a,b}\cap[\Delta^n(a,\varepsilon_1)\times\partial\Delta(b,\rho)]=\emptyset$, it follows from Theorem 1.3 that $S_{a,b}$ is pluripolar.

(ii) Let $V_o\subset\subset V_1\subset\subset V$ be domains with $V_o\cap F^*\neq\emptyset$. From (i), there are $\varepsilon>0$ and a pluripolar set S_a such that $[\Delta^n(a,\varepsilon)\times V_1]\setminus S_a\subset\widehat{G}$ for every $a\in E^*$.

Let $b\in\bar{V_o}$ and $c\in F^*\cap V_o$. Since for every $z\in U$ we can find $\varepsilon>0$, $\varsigma>0$ and S_z such that $\Delta^n(z,\varepsilon)\times\Delta(c,\varsigma)\setminus S_z\subset\widehat{G}$, then $R_{f,c}(z)>0$ and by Lemma 2.3, $-\log R_{f,c}\in Psh(D)$.

Let $w_1,...,w_k$ in V_1 and $r_1,...,r_k$ in $\mathbb{R}^*_+$ such that:

1) $w_1=c$, $w_k=b$ and $\bigcup_{i=1}^k\Delta(w_i,r_i)\subset V_1$.

2) $w_{i+1}\in\Delta(w_i,r_i)$.

Put $v_1=-\log R_{f,c}+\log r_1$; we have $v_1\leq 0$ on E^*, so by Lemma 2.4, $v\leq 0$ on E. By definition of $R_{f,c}$ there are an open neighborhood $\Omega(w_1)$ of E and a pluripolar set $S(w_1)$ such that $[\Omega(w_1)\times\Delta(w_1,r_1)]\setminus S(w_1)\subset\widehat{G}$. Since $w_2\in\Delta(w_1,r_1)$, then $-\log R_{f,w_2}\in Psh(\Omega(w_1))$. Put $v_2=-\log R_{f,w_2}+\log r_2$, again by Lemma 2.4 we obtain an open neighborhood $\Omega(w_2)$ of E and a pluripolar $S(w_2)$ such that $[\Omega(w_2)\times\Delta(w_2,r)]\setminus S(w_2)\subset\widehat{G}$. After a finite number of steps we can construct an open neighborhood $\Omega(w_k)$ of E and a pluripolar set $S(w_k)$ such that $\Omega(w_k)\times\Delta(b=w_k,r_k)\setminus S(w_k)\subset\widehat{G}$.

From above, we conclude the existence of an open neighborhood Ω_o of E and a pluripolar set S_o satisfying $[\Omega_o\times V_o]\setminus S_o\subset\widehat{G}$. Now, since V_o is arbitrary, we can obtain the result by taking a countable sequence $\{V_i\}_{i\geq 1}$ such that $V_i\subset\subset V_{i+1}$ and $\bigcup_{i\geq 1}V_i=V$.

Suppose now $m>1$: Let $a\in E^*$, $b\in V$ and $r>0$ such that $\Delta^m(b,r)=\Delta_1(b_1,r)\times\Pi_{i=2}^m\Delta(b_i,r)\subset V$. Take $\alpha\in\Delta_1(b_1,r)$ and $\beta\in\Pi_{i=2}^m\Delta(b_i,r)$ with $u(a,\alpha,\beta)\neq-\infty$. The set $A_{a,\beta}=\{t\in\Delta_1(b_1,r)\,/\,u(a,t,\beta)=-\infty\}$ is polar. Put $A=\Delta(b_1,r)\backslash B$, where B is the set of points $t\in\Delta_1(b_1,r)$ such that $(a,t,\beta)\in\widehat{G}$. Since $A\subset A_{a,\beta}$, A is polar and closed. It follows that there is $\rho\in]0,r[$ such that the circle $S(b_1,\rho)$ does not intersect A, hence $\{a\}\times S(b_1,\rho)\times\{\beta\}\subset\widehat{G}$. This implies that there are ε_1, $\eta>0$ and an annulus $C_{R_1}^{R_2}$ centered at b with $\rho\in]R_1,R_2[$ such that

$$\Delta^n(a,\varepsilon_1)\times C_{R_1}^{R_2}\times\Delta^{m-1}(\beta,\eta)\subset\widehat{G}.$$

Put $L:=E^*\times\Delta_1$. Observe that $\{a\}\times S(b_1,\rho)\subset L$.

Let $t\in S(b_1,\rho)$ and $\varsigma>0$ such that $\Delta(t,\varsigma)\subset C_{R_1}^{R_2}$. Put

$$Y:=[\Delta^n(a,\varepsilon_1)\times\Delta(t,\varsigma)]\times\Delta^{m-1}(\beta,\eta)\cup L\times\Pi_{i=2}^m\Delta_i(b_i,r).$$

Let $h\in\mathcal{O}(\widehat{G})$ with $\widehat{G}$ being the envelope of holomorphy of h. Obviously $h\in\mathcal{O}_s(Y\setminus P)$, thus by Lemma 2.2, h can be extended to an open neighborhood $\widehat{H}$ of $Y^*\setminus P$. Thus $\widehat{H}\subset\widehat{G}$.

By induction, there are a pluripolar set $Z_{a,t}$, $r_o>0$ and $\rho_o>0$ such that

$$[\Delta^n(a,r_o)\times\Delta(t,r_o)\times\Pi_{i=2}^m\Delta_i(b_i,\rho_o)]\setminus Z_{a,t}\subset\widehat{H}\subset\widehat{G}.$$

For a finite number of values of t, we can find $R_1<R_3<\rho<R_4<R_2$, ε, ρ' and a pluripolar set ${S'}_{a,b}$ such that:

$$\Delta^n(a,\varepsilon) \times C_{R_3}^{R_4} \times \Pi_{i=2}^m \Delta(b_i,\rho') \setminus S'_{a,b} \subset \widehat{G}.$$

Put $\Sigma := \Delta^n(a,\varepsilon) \times \Delta_1(b,R_3) \times \Pi_{i=2}^m \Delta(b_i,\rho') \cap \widehat{G}$ and $S''_{a,b} = \Delta^n(a,\varepsilon) \times \Delta_1(b_1,R_3) \times \Pi_{i=2}^m \Delta(b_i,\rho') \setminus \Sigma$. Obviously $S''_{a,b}$ is pseudoconcave and $\overline{S''}_{a,b} \cap (\Delta^n(a,\varepsilon) \times \partial\Delta_1(b_1,\rho) \times \Pi_{i=2}^m \Delta_i(b_i,\rho')) = \emptyset$. Also for all $(z,t) \in L^* \cap \Delta^n(a,\varepsilon) \times \Delta(b_1,\rho)$, we have $S''_{a,b}(z,t) \subset P_{z,t}$, so $S''_{a,b}(z,t)$ is pluripolar. It follows that $S''_{a,b}$ is pluripolar. We conclude that there are a pluripolar set $S_{a,b} = S'_{a,b} \cup S''_{a,b}$, and the numbers ε and ρ such that $(\Delta^n(a,\varepsilon) \times \Delta^m(b,\rho)) \setminus S_a \subset \widehat{G}$.

Using the same techniques as in (ii) in the case $m = 1$, we can construct an open neighborhood Ω_1 of $E \times V$ and a pluripolar set S_1 such that $\Omega_1 \setminus S_1 \subset \widehat{G}$.

Similarly, we construct an open neighborhood Ω_2 of $U \times F$ and a pluripolar set S_2 such that $\Omega_2 \setminus S_2 \subset \widehat{G}$.

Put now $\Omega = \Omega_1 \cup \Omega_2$ and $S' = S_1 \cup S_2$, then $\Omega \setminus S' \subset \widehat{G}$. Let $S = S' \setminus S_o$, where S_o is the set of points of S' in which $\widehat{f}$ is holomorphic, the set S is closed and pluripolar in Ω and $\Omega \setminus S \subset \widehat{G}$.

It is easy to see that $S \cap \mathcal{X}^* \subset P$ and then $\widehat{f}_{|\mathcal{X}^* \setminus P} = f_{|\mathcal{X}^* \setminus P}$. Thus the theorem is proved. ■

In the following result we combine Theorem 1.1 with Theorem 1.4:

Theorem 2.5. *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be two pseudoconvex domains and $E \subset U$ and $F \subset V$ be locally pluriregular sets and P a closed pluripolar set in an open neighborhood W of $\mathcal{X} := E \times V \cup U \times F$. Then for every function $f \in \mathcal{O}_s(\mathcal{X} \setminus P)$, there exist a closed pluripolar set $\widehat{S}$ of $\widehat{\mathcal{X}}$ and a unique function $h \in \mathcal{O}(\widehat{\mathcal{X}} \setminus \widehat{S})$ such that $\widehat{S} \cap \mathcal{X}^* \subset P$ and $h_{|\mathcal{X}^* \setminus P} = f_{|\mathcal{X}^* \setminus P}$.*

Proof. By Theorem 1.1, there are an open neighborhood Ω of $\mathcal{X}$, a closed pluripolar set S in Ω and a function $\widehat{f} \in \mathcal{O}(\Omega \setminus S)$ such that $S \cap \mathcal{X}^* \subset P$ and $\widehat{f}_{|\mathcal{X}^* \setminus P} = f_{|\mathcal{X}^* \setminus P}$. Since $\mathcal{X} \subset \Omega$, we have $\widehat{\mathcal{X}} \subset \widehat{\Omega}$, where $\widehat{\Omega}$ is the envelope of holomorphy of Ω. Otherwise by Theorem 1.4, there is a closed pluripolar set $\widehat{S}$ in $\widehat{\Omega}$ such that $\widehat{S} \cap \Omega \subset S$ and $\widehat{\Omega} \setminus \widehat{S}$ is the envelope of holomorphy of $\Omega \setminus S$. It then follows that there is $h \in \mathcal{O}(\widehat{\mathcal{X}} \setminus \widehat{S})$ such that $h_{|\mathcal{X}^* \setminus P} = f_{|\mathcal{X}^* \setminus P}$. The uniqueness of h follows from the fact that $\mathcal{X}^* \setminus P$ is non-pluripolar. ■

References

1. O. Alehyane and A. Zeriahi, Une nouvelle version du théorème de Hartogs pour les applications séparément holomorphes entre espaces analytiques, *Ann. Pol. Math.* **76** (2001) 245–278.
2. E. M. Chirka, The extension of pluripolar singularity sets, *Proc. Steklov Inst. Math.* **200** (1993) 369–373.
3. E. M. Chirka and A. Sadullaev, On the continuation of functions with polar singularities, *Math. USSR. Sb.* **60** (1988) 377–384.

4. A. Edigrian and E. A. Poletsky, Product property of the relative extremal function, *Bull. Polon. Acad. Sci. Math.* **45** (1995) 331–335.
5. M. Jarnicki and P. Pflug, Cross theorem, *Ann. Pol. Math.* **77** (2001) 295–302.
6. M. Jarnicki and P. Pflug, *An Extension Theorem for Separately Holomorphic Functions with Singularities*, IMUJ Preprint 2001/27.
7. M. Jarnicki and P. Pflug, *An extension theorem for separatly holomorphic functions with pluripolar singularities*, www. arXiv:math.cv/0112082 v1.
8. O. Oktem, Extension of separatly analytic functions and applications to range characterization of exponential Radon transform, *Ann. Polon. Math.* **70** (1998) 195–213.
9. O. Oktem, Extension separately analytic functions in $\mathbb{C}^{n+m}$ with singularities, Extension of separetely functions and applications to mathematical tomography, Ph. D. Thesis, Dep. Math. Stockholm Univ., 1999.
10. Nguyen Thanh Van, Separate analyticity and related subjects, *Vietnam J. Math.* **25** (1997) 81–90.
11. Nguyen Thanh Van and A. Zeriahi, Une extension du théorème de Hartogs sur les fonctions séparément analytiques, *Analyse Complexe multivariable: Recents Developpements,* (28 Mars - 3 Avril 1988), Alex Meril ed., 183-194.
12. A. Sadullaev, Plurisubharmonic functions, several complex variables II, geometric function theory, *Encycl. Math. Sci.* **8** (1989) 59–106.
13. B. Shiffman, On separate analyticity and Hartogs theorem, *Indiana Univ. Math. J.* **38** (1989) 943–957.
14. J. Siciak, Analyticity and separate analyticity of functions defined on lower dimensional subsets of $\mathbb{C}^n$, *Zeszyty Nauk. Jagiellon. Prace Mat.* **13** (1969) 53–70.
15. J. Siciak, Separately analytic functions and envelopes of holomorphy of some lower dimensional subsets of $\mathbb{C}^n$, *Ann. Pol. Math.* **22** (1970) 145–171.
16. J. Siciak, Holomorphic functions with singularities on algebraic sets, *Univ. Iagel. Acta Math.* **39** (2001) 9–16.
17. V. P. Zahariuta, Separately holomorphic functions, generalization of Hartogs theorem and envelopes of holomorphy, *Math. USSR. Sb.* **30** (1976) 51–76.