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On the Weak Law of Large Numbers for *d*-Dimensional Arrays in Von Neumann Algebra

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Abstract. We investigate the weak law of large numbers for d-dimensional arrays of pairwise independent measurable operators. Some related results are considered.

1. Introduction and Notations

The weak law of large numbers in von Neumann algebra was considered by some authors. In [7] this law have been proved for martingale differences and quadratic forms. In [2, 5] some results were obtained for independent sequences of measurable operators. Note that the same problem for multidimensional arrays is not yet studied.

The aim of this paper is to give the weak law of large numbers for d-dimensional arrays of pairwise independent measurable operators. Our results derive some results in [2, 5] as corollaries, and can be viewed as non-commutative extensions of some results of [1, 3, 4].

Let us begin with some definitions and notations. Throughout this paper, $\mathcal{A}$ denotes a von Neumann algebra with faithfull normal tracial state τ; $\tilde{\mathcal{A}}$ denotes the algebra of measurable operators in Segal-Nelson's sence (see [2]).

Let W_1 and W_2 be two von Neumann subalgebras of $\mathcal{A}$. W_1 is said to be independent of W_2 if for all $X \in W_1$ and $Y \in W_2$

$$\tau(X.Y) = \tau(X).\tau(Y).$$

Two elements X, Y in $\tilde{\mathcal{A}}$ are said to be independent if the von Neumann algebras $W(X)$ and $W(Y)$ generated by X and Y, respectively, are independent.

Let $\mathbb{N}^d = \{\overline{n} = (n_1, n_2, \cdots, n_d), n_i \in \mathbb{N}, i = 1, ..., d\}$ (where $d \geq 1$ is fixed integer). $\mathbb{N}^d$ is partially ordered by agreeing that

$$\overline{k} = (k_1, k_2, \cdots, k_d) \leq \overline{m} = (m_1, m_2, \cdots, m_d) \quad \text{if} \quad k_i \leq m_i, \;\; i = 1, ..., d.$$

For $\overline{n} = (n_1, n_2, \cdots, n_d)$, we put

$$|\overline{n}| = n_1.n_2. \cdots .n_d = \text{ card } \{\overline{k} \in N^d; \overline{k} \leq \overline{n}\}.$$

An array $(X(\overline{n})\overline{n} \in \mathbb{N}^d) \subset \tilde{\mathcal{A}}$ is said to be the array of pairwise independent elements if for all $\overline{m}, \overline{n} \in \mathbb{N}^d; \overline{m} \neq \overline{n}$; $X(\overline{m})$ and $X(\overline{n})$ are independent. For each self-adjoint element X in $\tilde{\mathcal{A}}$ we denote by $e_\Delta(X)$ the spectral projection of X corresponding to the Borel subset Δ of the real line $\mathbb{R}$.

Two self- adjoint elements X and Y of $\tilde{\mathcal{A}}$ are said to be identically distributed if $\tau(e_\Delta(X)) = \tau(e_\Delta(Y))$ for all Borel subset $\Delta \subset \mathbb{R}$.

For further information we refer to [2, 6, 9].

2. The Weak Law of Large Numbers for Multidimensional Arrays

In this section we will consider the convergence in measure of d-dimensional arrays of measurable operators as $|\overline{n}| \to \infty$. Note that the limit as $|\overline{n}| \to \infty$ is stronger than the limit as all $\overline{n}_i \to \infty$; $(i = 1, ..., d)$ (see [8]).

Lemma 2.1. *Let $(X(\overline{n}),\ \overline{n} \in \mathbb{N}^d)$ be an array of measusable operators. The following conditions are equivalent:*

i) *$X(\overline{n}) \to 0$ in measure as $|\overline{n}| \to \infty$;*

ii) *For each $\epsilon > 0$, there exist an array $(p(\bar{n});\ \bar{n} \in \mathbb{N}^d \subset Proj\mathcal{A}$ and $n_0 \in \mathbb{N}$ such that $X(\overline{n})p(\overline{n}) \in \mathcal{A}$, $\|X(\overline{n})p(\overline{n})\| < \epsilon$ for all $\overline{n} \in \mathbb{N}^d$; $|\overline{n}| \geq n_0$;*

iii) *For each $\epsilon > 0$, $\tau(e_{[\epsilon,\infty)}(|X(\overline{n})|)) \to 0$ as $|\overline{n}| \to \infty$.*

This lemma can be proved by the same techniques as in the case $d = 1$ (see [2], A48), so we omit the proof.

The main result of this section is the following theorem.

Theorem 2.2. *Let $(X(\overline{n})\ \overline{n} \in \mathbb{N}^d) \subset \tilde{\mathcal{A}}$ be an array of pairwise independent measurable operators, $0 < t_i < \infty (i = 1, ..., d)$; $\overline{n}(\overline{t}) = n_1^{1/t_1}.n_2^{1/t_2} \cdots n_d^{1/t_d}$. Suppose that*

$$\lim_{|\bar{n}| \to \infty} \sum_{\overline{k} \leq \overline{n}} \tau(e_{[\overline{n}(\overline{t}),\infty)}(|X(\overline{k})|)) = 0, \tag{2.1}$$

$$\lim_{|\overline{n}| \to \infty} \overline{n}(\overline{t})^{-1} \sum_{\overline{k} \leq \overline{n}} \tau(|X(\overline{k})| e_{[0,\overline{n}(\overline{t}))}(|X(\overline{k})|)) = 0, \tag{2.2}$$

$$\lim_{|\overline{n}| \to \infty} \overline{n}(\overline{t})^{-2} \sum_{\overline{k} \leq \overline{n}} \{\tau(X^2(\overline{k}) e_{[0,\overline{n}(\overline{t}))}(|X(\overline{k})|)) - |\tau(X(\overline{k}) e_{[0,\overline{n}(\overline{t}))}(|X(\overline{k})|))|^2\} = 0. \tag{2.3}$$

Then

$$\overline{n}(\overline{t})^{-1} \sum_{\overline{k} \leq \overline{n}} X(\overline{k}) \to 0 \tag{2.4}$$

in measure as $|\overline{n}| \to \infty$.

Proof. Put

$$\begin{aligned} S(\overline{n}) &= \sum_{\overline{k}\le\overline{n}} X(\overline{k}), \\ X^{(\overline{n})}(\overline{k}) &= X(\overline{k}).e_{[0,\overline{n}(\overline{t}))}(|X(\overline{k})|), \\ \overline{S}(\overline{n}) &= \sum_{\overline{k}\le\overline{n}} X^{(\overline{n})}(\overline{k}), \\ \overline{m}(\overline{n}) &= \tau(\overline{S}(\overline{n})) \ = \sum_{\overline{k}\le\overline{n}} \tau(X^{(\overline{n})}(\overline{k})). \end{aligned}$$

Using similar technique as in Theorem 3.1 of [5] we have, for an arbitrary $\gamma\ > 0$

$$p = e_{[2\gamma,\infty)}(|S(\overline{n}) - \overline{m}(\overline{n})|) \wedge e_{[0,\gamma)}(|\overline{S}(\overline{n}) - \overline{m}(\overline{n})|) \wedge (\bigwedge_{\overline{k}\le\overline{n}} e_{[0,\overline{n}(\overline{t}))}(|X(\overline{k})|)) = 0.$$

This implies

$$\begin{aligned} e_{[2\gamma,\infty)}(|S(\overline{n}) - \overline{m}(\overline{n})|) &\le \big[e_{[0,\gamma)}(|\overline{S}(\overline{n}) - \overline{m}(\overline{n})|) \wedge (\wedge_{\overline{k}\le\overline{n}} e_{[0,\overline{n}(\overline{t}))}(|X(\overline{k})|)\big]^{\perp} \\ &= e_{[\gamma,\infty)}(|\bar{S}(\overline{n}) - \overline{m}(\overline{n})|) \vee \big(\bigvee_{\overline{k}\le\overline{n}} (e_{[0.\overline{n}(\overline{t})}(|X(\overline{k})|))^{\perp}\big). \end{aligned} \tag{2.5}$$

Using Tchebyshev's inequality, the pairwise independence of elements $X(\overline{n})$ ($\overline{n} \in \mathbb{N}^d$) and the equality

$$\tau(|X - \tau(X)|^2) = \tau(|X|^2) - |\tau(X)|^2, \quad \forall X \in L^2(\mathcal{A}, \tau),$$

we get

$$\begin{aligned} &\tau(e_{[\gamma,\infty)}(|\overline{S}(\bar{n}) - \overline{m}(\overline{n})|)) \le \gamma^{-2}\tau(|\overline{S}(\overline{n}) - \overline{m}(\overline{n})|^2) \\ &= \gamma^{-2}\tau(|\sum_{\overline{k}\le\overline{n}}(X^{(\overline{n})}(\overline{k}) - \tau(X^{(\overline{n})}(\overline{k})))\big|^2) \\ &= \gamma^{-2}\tau\big[\big(\sum_{\overline{k}\le\overline{n}}(X^{(\overline{n})}(\overline{k}) - \tau(X^{(\overline{n})}(\overline{k})))\big)^*\big(\sum_{\overline{k}\le\overline{n}}(X^{(\overline{n})}(\overline{k}) - \tau(X^{(\overline{n})}(\overline{k})))\big)\big] \\ &= \gamma^{-2}\sum_{\overline{k}\le\overline{n}} \tau(|X^{(\overline{n})}(\overline{k}) - \tau\big(X^{(\overline{n})}(\overline{k})\big)|^2\big) \\ &= \gamma^{-2}\sum_{\overline{k}\le\overline{n}} \{\tau(|X^{(\overline{n})}(\overline{k})|^2\big) - |\tau(X^{(\overline{n})}(\overline{k}))\big|^2\}. \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6) we obtain

$$\begin{aligned} \tau\big(e_{[2\gamma,\infty)}|S(\overline{n}) - \overline{m}(\overline{n})|\big) &\le \gamma^{-2}\sum_{\overline{k}\le\overline{n}} \big\{\tau(|X^{(\overline{n})}(\overline{k})|^2\big) - |\tau(X^{(\overline{n})}(\overline{k}))\big|^2\big\} \\ &\quad + \sum_{\overline{k}\le\overline{n}} \tau(e_{[\overline{n}(\overline{t}),\infty)}(|X(\overline{k})|)). \end{aligned}$$

Now, let $\epsilon > 0$ be given. Put $\gamma = \frac{\overline{n}(\overline{t})\epsilon}{2}$, we have

$$\tau\big(e_{[\epsilon,\infty)}|\overline{n}(\overline{t})^{-1}(S(\overline{n}) - \overline{m}(\overline{n}))|\big) = \tau\big(e_{[\overline{n}(\overline{t}).\epsilon,\infty)}(|S(\overline{n}) - \overline{m}(\overline{n})|)\big)$$
$$\leq \frac{4}{\epsilon^2.\overline{n}(\overline{t})^2}\sum_{\overline{k}\leq\overline{n}}\big\{\tau(|X^{(\overline{n})}(\overline{k})|^2) - |\tau(X^{(\overline{n})}(\overline{k}))|^2\big\} + \sum_{\overline{k}\leq\overline{n}}\tau(e_{[\overline{n}(\overline{t}),\infty)}(|X(\overline{k})|)).$$

By (2.1) and (2.3) we obtain

$$\tau\big(e_{[\epsilon,\infty)})(|\overline{n}(\overline{t})^{-1}(S(\overline{n}) - \overline{m}(\overline{n}))|\big) \to 0$$

as $|\overline{n}| \to \infty$.

This together with Lemma 2.1 implies

$$\overline{n}(\overline{t})^{-1}\big(S(\overline{n}) - \overline{m}(\overline{n})\big) \to 0 \tag{2.7}$$

in measure as $|\overline{n}| \to 0$.

From (2.7) and (2.2) we get (2.4), which completes the proof. ∎

Let us note that Theorem 2.2 implies Theorem 3.2 of [5] as a corollary for $d = 1,\ t_1 = 1$. In this case, it is also a non-commutative analogue of the sufficient condition of a well- known theorem in classical probability (see Theorem A of [4, p. 290]), but the proof here is quite different.

The following corollaries are the non-commutative analogues of some theorems of [1] and [3].

Corollary 2.3. *Let* $(X(\overline{n})\ \overline{n} \in N^d)$ *be an array of self- adjoint pairwise independent identically symmetrically distributed elements of* $\tilde{\mathcal{A}}$. *Suppose that*

$$\lim_{|\overline{n}|\to\infty} |\overline{n}|.\tau(e_{[n_1^{1/t_1}\cdots n_d^{1/t_d},\infty)}(|X(\overline{1})|)) = 0, \tag{2.8}$$
$$\lim_{|\overline{n}|\to\infty} n_1^{1-2/t_1}\cdots n_d^{1-2/t_d}\tau(X^2(\overline{1})e_{[0,n_1^{1/t_1}\cdots n_d^{1/t_d}]}(|X(\overline{1})|)) = 0. \tag{2.9}$$

Then

$$n_1^{-1/t_1}\cdots n_d^{-1/n_d}\sum_{\overline{k}\leq\overline{n}} X(\overline{k}) \to 0$$

in measure as $|\overline{n}| \to \infty$.

(*where* $0 < t_i < \infty$ $(i = \overline{1,d})$ $\overline{1} = (1, 1, \cdots, 1) \in \mathbb{N}^d$).

Proof. By the symmetric distribution of elements $X(\overline{n})$ $(\overline{n} \in \mathbb{N}^d)$ we have

$$\tau(e_{[0,n_1^{1/t_1}\cdots n_d^{1/t_d})}(|X(\overline{k})|)) = 0, \quad (\forall\overline{k} \in \mathbb{N}^d),$$

hence (2.2) is satisfied immediately.

Moreover, by (2.8) (2.9) and the assumption that all $X(\overline{n})$ $(\overline{n} \in \mathbb{N}^d)$ are identically distributed, the conditions (2.1) and (2.3) are also satisfied.

Thus the array $(X(\overline{n})\ \ \overline{n} \in \mathbb{N}^d)$ satisfies all assumptions of Theorem 2.2 and the corollary is proved. ∎

Corollary 2.4. (Weak law of large numbers) *Let* $(X(\overline{n})\ \overline{n} \in \mathbb{N}^d)$ *be an array of self-adjont pairwise independent and identically distributed elements of* $\mathcal{A}$. *If* $\tau(|X(\overline{1})|) < \infty$ *then*

$$|\overline{n}|^{-1} \sum_{\overline{k} \leq \overline{n}} X(\overline{k}) \to \tau(X(\overline{1}))$$

in measure as $|\overline{n}| \to \infty$.

This corollary can be proved easily and we omit the proof. ■

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