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Existence of Periodic Solutions and Bounded Invariant Sets for Non-Autonomous Third-Order Differential Equations

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Abstract. In this paper we consider the third-order non-autonomous differential equation

$$\dddot{x} + \omega^2 \dot{x} = \mu F(x, \dot{x}, \ddot{x}, t, \epsilon) \tag{1}$$

where the C^r $(r \geq 2)$ map F is periodic in t and μ, ϵ are real parameters. We give some conditions for Eq. (1) in order to reduce it to a second-order differential equation. Then we show that the periodic (homoclinic) solutions of the second-order equation imply the periodic (homoclinic) solutions of Eq. (1). Then we use the Hopf bifurcation theorem for the second-order equation and obtain periodic solutions and non-trivial bounded invariant sets for it. The effect of the quadratic terms on Eq. (1) is also studied.

1. Introduction

There are many papers on the existence of periodic solutions for second-order differential equations (for example see [1, 2, 5]), but there are not many papers for nonlinear third-order differential equations. These kinds of equations arise in engineering, for example, in problems related to energy and acceleration. Because of the topological characteristics of the three-dimensional space the investigation of periodic solutions for the third-order differential equations is a difficult problem.

In [6], Mehri and Niksirat considered the nonlinear equation

$$\dddot{X} = F(X, \dot{X}, \ddot{X}) \quad X \in \mathbb{R}^{2n+1}$$

and showed that under some conditions, there exists $\omega > 0$ such that $\dot{X}(\omega) = \dot{X}(0)$. Then Mehri and Niksirat [7] considered the nonlinear third-order differential equation

$$\dot{\ddot{x}} + \omega^2 \dot{x} = \mu F(x, \dot{x}, \ddot{x}) \tag{2}$$

and obtained some conditions for the existence of a periodic solution for it. We will write their result in the following form. Consider Eq. (2) where F is assumed to be smooth enough such that the existence and uniqueness of the solution are guaranteed. Let $\lambda = 2\pi\omega^{-1} + \mu\tau$ and define

$$\Gamma_\lambda(a, b, \tau) = \begin{pmatrix} x(0) - x(\lambda) \\ \dot{x}(0) - \dot{x}(\lambda) \\ \ddot{x}(0) - \ddot{x}(\lambda) \end{pmatrix}.$$

Assume that there exist $a_0, b_0 \neq 0$ such that

$$\int_0^{2\pi\omega^{-1}} F(a_0 + b_0 \cos\omega t, -b_0 \sin\omega t, -b_0 \cos\omega t)\, dt = 0$$

$$\int_0^{2\pi\omega^{-1}} F(a_0 + b_0 \cos\omega t, -b_0 \sin\omega t, -b_0 \cos\omega t)\, \cos\omega t\, dt = 0$$

$$\int_0^{2\pi\omega^{-1}} \int_0^{2\pi\omega^{-1}} [h(t)F_x(u) - F_x(t)h(u)] \cos\omega t\, dt\, du \neq 0$$

where

$$\begin{aligned} F_v(s) &= \frac{\partial F}{\partial v}((a_0 + b_0 \cos\omega s, -b_0 \sin\omega s, -b_0 \cos\omega s) \quad v \in \{x, \dot{x}, \ddot{x}\} \\ h(s) &= \cos\omega s F_x(s) - \sin\omega s F_{\dot{x}}(s) - \cos\omega s F_{\ddot{x}}(s). \end{aligned}$$

Then there exists $\lambda_0 > 0$ such that for all $|\lambda| < \lambda_0$ Eq. (2) has a nonconstant λ-periodic solution. Also they proved that if all the stationary points of Eq. (2) are non-periodic and $\Gamma_0(a, b, \tau)$ has odd signed periodic orbits then Eq. (2) has at least one signed periodic solution for $0 \leq \mu \leq 1$. In [10], Rabiei and Afsharnejad considered the Eq. (2). They showed that if the map F is the sum of partial derivatives of a map $f(x, \dot{x})$ with $f_{xy}(0, 0) \neq 0$, then Eq. (2) has many periodic solutions.

In what follows, we extend the concept of [10] for non-autonomous third-order equations. We consider the third-order differential equation

$$\dot{\ddot{x}} + \omega^2 \dot{x} = \mu F(x, \dot{x}, \ddot{x}, t, \varepsilon) \quad \varepsilon > 0 \tag{3}$$

and obtain some conditions for the existence of periodic solutions for it. These conditions are applicable to many third-order differential equations. Let $H(x, \dot{x}, t, \varepsilon) = f(x, \dot{x}) + \varepsilon g(x, \dot{x}, t)$ be $C^r (r \geq 3)$ and

$$F(x,\dot{x},\ddot{x},t,\varepsilon) = H_x(x,\dot{x},t,\varepsilon)\dot{x} + H_{\dot{x}}(x,\dot{x},t,\varepsilon)\ddot{x} + H_t(x,\dot{x},t,\varepsilon). \qquad (4)$$

We show that if $f_{x\dot{x}}(0,0) \neq 0$, then under some conditions, Eq. (3) has many periodic solutions.

In Sec. 2 we present some mathematical preliminaries and show that the existence of periodic (homoclinic) solutions for the second-order differential equation implies the existence of periodic (homoclinic) solutions for the third-order differential equation. The existence of the fixed points and the type of these fixed points is studied in Sec. 3. We also use the averaging theorem and find periodic solutions for the third-order differential equation. Sec. 4 is devoted to the existence of non-trivial bounded invariant sets for the third-order differential equation. In Sec. 5 we study the effect of the quadratic terms on Eq. (3).

2. Preliminaries

Our main tools in the development of the article are the averaging method and the Poincare-Andronove-Hopf 's bifurcation theorem which is usually called Hopf bifurcation theorem. The averaging methods used for a non-autonomous differential equation are very various; the method which we will use is usually called time averaging method. Using the Sec. 5.1 of [3], we explain the time averaging method.

Consider the system

$$\dot{x} = \varepsilon f(x,t,\varepsilon) \quad x \in U \subset \mathbb{R}^n, \quad 0 < \varepsilon << 1 \qquad (5)$$

where $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ is C^r, $r \geq 0$, bounded on bounded sets, and of period $T > 0$ in t. The time averaged system of Eq. (5) is defined by

$$\dot{y} = \varepsilon \frac{1}{T} \int_0^T f(y,t,0)dt = \varepsilon \widetilde{f}(y). \qquad (6)$$

In this situation we have

Theorem 1. [3] *There exists a C^r change of coordinates $x = y + \varepsilon W(y,t,\varepsilon)$ under which Eq.* (5) *becomes*

$$\dot{y} = \varepsilon \widetilde{f}(y) + \varepsilon^2 f_1(y,t,\varepsilon)$$

where f_1 is of period T in t. Moreover

(i) *if p_0 is a hyperbolic fixed point of Eq.* (6) *then for ε sufficiently small, Eq.* (5) *has a unique hyperbolic periodic orbit $\gamma_\varepsilon(t) = p_0 + \mathcal{O}(\varepsilon)$ of the same stability type as p_0.*

(ii) *Consider the equations*

$$(iia) \begin{cases} \dot{y} = \varepsilon \widetilde{f}(y) \\ \dot{\theta} = 1 \end{cases} \qquad (iib) \begin{cases} \dot{y} = \varepsilon \widetilde{f}(y) + \varepsilon^2 f_1(y,\theta,\varepsilon) \\ \dot{\theta} = 1 \end{cases}$$

If Eq. (*iia*) *has a hyperbolic periodic solution γ_0, then for ε sufficiently small Eq.* (*iib*) *has a hyperbolic invariant torus $T_\varepsilon = \gamma_0 \times S^1 + \mathcal{O}(\varepsilon)$.*

Proof. See [3 Secs. 4.1 & 4.4] .

There are several versions of the Hopf bifurcation theorem but we give the version which is in [12]; the interested reader can find other versions of this theorem in [4] or [3].

Theorem 2. (Hopf bifurcation theorem) *Consider the* C^r $(r \geq 3)$ *planner vector field*

$$\dot{X} = F(X, \lambda) \quad \lambda \in \mathbb{R}. \tag{7}$$

Assume that there exist an open interval I *and a* C^1 *map* $(x, y) : I \to R$ *such that*

$$F(x(\lambda), y(\lambda), \lambda) = 0.$$

Also assume that the linear part of Eq. (7) *has the eigenvalues* $\alpha(\lambda)^+_- i\beta(\lambda)$ *with* $\alpha(\lambda_0) = 0$ *and* $\beta(\lambda_0) \neq 0$. *If* $\alpha'(\lambda_0) \neq 0$ *then there exist a unique three-dimensional center manifold passing through* $(x(\lambda_0), y(\lambda_0), \lambda_0)$ *in* $\mathbb{R}^2 \times \mathbb{R}$ *and a smooth system of coordinates for which Eq.* (7) *in polar coordinates is given by*

$$\begin{cases} \dot{r} = (\alpha'(\lambda_0)\lambda + \rho r^2)r + \mathcal{O}(\lambda^2 r, \lambda r^3, r^5) \\ \dot{\theta} = \beta(\lambda_0) + \beta'(\lambda_0)\lambda + br^2 + \mathcal{O}(\lambda^2, \lambda r^2, r^3). \end{cases} \tag{8}$$

Furthermore, if $\rho \neq 0$, *then for each* $\lambda \neq 0$ *with* $\lambda\alpha'(\lambda_0)\rho < 0$ *and* $|\lambda|$ *sufficiently small, there is a surface of hyperbolic periodic solutions in the center manifold. Moreover if* $\rho < 0$, *these periodic solutions are stable and if* $\rho > 0$, *these periodic solutions are repelling. In addition, if we consider* (7) *in its standard form, i.e*

$$\begin{cases} \dot{x} = \alpha(\lambda)x - \beta(\lambda)y + f(x, y, \lambda) \\ \dot{y} = \beta(\lambda)x + \alpha(\lambda)y + g(x, y, \lambda) \end{cases}$$

then

$$\begin{aligned} \rho = & \frac{1}{16}\left[f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}\right] \\ & + \frac{1}{16\beta(\lambda_0)}\left[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\right] \end{aligned}$$

where all partial derivatives are computed at $(x(\lambda_0), y(\lambda_0), \lambda_0))$.

Proof. See [12, Sec. 3.1B].

Remark 1. In the Hopf bifurcation theorem, if $\rho = 0$ then the periodic solutions exist. In fact for each $\varepsilon > 0$ and any neighborhood U of $x(\lambda_0)$, there exists λ with $|\lambda - \lambda_0| < \varepsilon$ such that the Eq. (7) has a non-trivial periodic solution in U. This fact is proven in [4, Sec. 11.2].

Definition 1. *Consider Eq.* (7) *and suppose that the Hopf bifurcation theorem holds for it. We say that for* $\lambda = \lambda_0$, *a Hopf bifurcation occurs at* $x(\lambda_0)$ *for Eq.* (7).

Now we reduce the third-order differential equation to a second-order differential equation. Consider the equation

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -\omega^2 y + \mu F(x, y, z, t, \varepsilon), \quad \mu, \ \varepsilon \in \mathbb{R} \end{cases} \tag{9}$$

where F is C^r, $r \geq 2$ and of period $2\pi\omega^{-1}$ in t. Assume that there exists the C^{r+1} map $H(x, \dot{x}, t, \varepsilon) = f(x, \dot{x}) + \varepsilon g(x, \dot{x}, t)$ such that

A_1 : $f(0, 0) = 0$ and $Df(0, 0) = 0$,

A_2 : $g(x, \dot{x}, t)$ is of period $2\pi\omega^{-1}$ in t,

A_3 : $F(x, \dot{x}, \ddot{x}, t, \varepsilon) = H_x(x, \dot{x}, t, \varepsilon)\dot{x} + H_{\dot{x}}(x, \dot{x}, t, \varepsilon)\ddot{x} + H_t(x, \dot{x}, t, \varepsilon)$.

Substituting A_3 in Eq. (3) we obtain

$$\begin{aligned} &\frac{d}{dt}(\ddot{x} + \omega^2 x - \mu H(x, \dot{x}, t, \varepsilon)) \\ &= \dddot{x} + \omega^2 \dot{x} - \mu(\underbrace{H_x(x, \dot{x}, t, \varepsilon)\dot{x} + H_{\dot{x}}(x, \dot{x}, t, \varepsilon)\ddot{x} + H_t(x, \dot{x}, t, \varepsilon)}_{F(x,\dot{x},\ddot{x},t,\varepsilon)}) = 0. \end{aligned}$$

Therefore

$$\ddot{x} + \omega^2 x - \mu H(x, \dot{x}, t, \varepsilon) = k$$

where k is a real constant. If we put $k = \omega^2\lambda$ and $\bar{x} = x - \lambda$ then we get

$$\ddot{\bar{x}} + \omega^2 \bar{x} = \mu H(\bar{x} + \lambda, \dot{\bar{x}}, t, \varepsilon).$$

Dropping the bars we obtain

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \mu[f(x + \lambda, y) + \varepsilon g(x + \lambda, y, t)]. \end{cases} \tag{10}$$

Lemma 1.

(i) *The curve $(x(t), \dot{x}(t))$ is a solution for Eq.* (10) *if and only if $(x(t) - \lambda, \dot{x}(t), \ddot{x}(t))$ is a solution for Eq.* (9)*. Furthermore, if $(x(t), \dot{x}(t))$ is a periodic (homoclinic) solution then $(x(t) - \lambda, \dot{x}(t), \ddot{x}(t))$ is a periodic (homoclinic) solution.*

(ii) *The non-trivial bounded invariant sets of Eq.* (10) *indicate the non-trivial bounded invariant sets of Eq.* (9)*.*

Proof.

(i) It is obvious that if $(x(t), \dot{x}(t))$ is a solution for Eq. (10) then $(x(t) - \lambda, \dot{x}(t), \ddot{x}(t))$ is a solution for Eq. (9). Let $(x(t), \dot{x}(t))$ be a periodic solution with period T_0. For each $t \in \mathbb{R}$ we have

$$\ddot{x}(t + T_0) = \lim_{h \to 0} \frac{\dot{x}(t + T_0 + h) - \dot{x}(t + T_0)}{h} = \lim_{h \to 0} \frac{\dot{x}(t + h) - \dot{x}(t)}{h} = \ddot{x}(t).$$

Therefore $(x(t)-\lambda, \dot{x}(t), \ddot{x}(t))$ is a periodic solution with period T_0. Let $(x(t), \dot{x}(t))$ a homoclinic solution with $\lim_{t \to \pm\infty}(x(t), \dot{x}(t)) = (x_0, 0)$. Since $(x_0, 0)$ is a fixed point for (10) so then $(x_0 - \lambda, 0, 0)$ is a fixed point for (9), moreover

$$\lim_{t\to\pm\infty} \ddot{x}(t) = \lim_{t\to\pm\infty} -\omega^2 x_0 + \mu H(x_0+\lambda, 0, t, \varepsilon) = 0.$$

Hence $(x(t)-\lambda, \dot{x}(t), \ddot{x}(t))$ is a homoclinic solution.
(ii) Let $0<\varepsilon$, μ, λ be fixed and let V be a non-trivial bounded invariant set for Eq. (10).

Assume that

$$\Psi(t,t_0,x_0,y_0) = \big(x(t,t_0,x_0,y_0), y(t,t_0,x_0,y_0)\big)$$

be the solution of Eq. (10) satisfying $\Psi(t_0,t_0,x_0,y_0) = (x_0,y_0)$. Since V is invariant so for each $t_0\in\mathbb{R}$ and $(x_0,y_0)\in V$, $\Psi(t,t_0,x_0,y_0)\in V \quad t\in\mathbb{R}$. We define

$$\widetilde{V}=\big\{\big(x(t,t_0,x_0,y_0)-\lambda,\ y(t,t_0,x_0,y_0),\ \dot{y}(t,t_0,x_0,y_0)\big) : t,\ t_0\in\mathbb{R}\ (x_0,y_0)\in V\big\}.$$

By (i), $\widetilde{V}$ is a non-trivial invariant set for Eq. (9). On the other hand

$$\dot{y}(t,t_0,x_0,y_0) = -\omega^2 x(t,t_0,x_0,y_0) + \mu\, H\big(x(t,t_0,x_0,y_0)+\lambda, y(t,t_0,x_0,y_0), t, \varepsilon\big).$$

Since V is a bounded invariant set and the map H is continuous and periodic in t, so $\dot{y}(t,t_0,x_0,y_0)$ is bounded. This shows that $\widetilde{V}$ is bounded. ■

3. Averaged Equation and Fixed Points

Consider Eq. (10). Changing the coordinates by $u=x(\varepsilon t)$, $v=y(\varepsilon t)$ we obtain

$$\begin{cases} \dot{u} = \varepsilon v \\ \dot{v} = \varepsilon\big(-\omega^2 u + \mu f(u+\lambda, v) + \varepsilon\mu g(u+\lambda, v, t)\big). \end{cases} \tag{11}$$

Averaging Eq. (11) we get

$$\begin{cases} \dot{u} = \varepsilon v \\ \dot{v} = \varepsilon\big(-\omega^2 u + \mu f(u+\lambda, v)\big). \end{cases}$$

Backing to the (x,y) coordinates we have

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \mu f(x+\lambda, y). \end{cases} \tag{12}$$

If $\lambda=0$ then the origin is a fixed point for Eq. (12). If $\lambda\neq 0$ then it is important for us to know fixed points of Eq. (12).

Theorem 3. *For each $0<M\in\mathbb{R}$ there exist an open rectangle $\mathcal{R}\subset\mathbb{R}^2$ containing $[-M,M]\times\{0\}$ and a C^1 map $x:\mathcal{R}\to R$ such that $(x(\mu,\lambda),0)$ is a fixed point for Eq.* (12). *Furthermore the real part of the eigenvalues of the linear part of Eq.* (12) *is $\mu f_y(x(\mu,\lambda)+\lambda, 0)$. Also for each $(\mu,0),(0,\lambda)\in\mathcal{R}$, $x(\mu,0)=x(0,\lambda)=0$.*

Proof. Consider the map

$$B : \mathbb{R}^3 \to \mathbb{R}$$
$$(x, \mu, \lambda) \to -\omega^2 x + \mu f(x + \lambda, 0).$$

Since for each $\mu \in \mathbb{R}$, $B(0, \mu, 0) = 0$ and $B_x(0, \mu, 0) = -\omega^2 \neq 0$, then by the implicit function theorem there exist an open neighborhood U_μ containing $(\mu, 0)$ and a C^1 map $x_\mu : U_\mu \to \mathbb{R}$ such that for each $(\nu, \lambda) \in U_\mu$, $B(x_\mu(\nu, \lambda)) = 0$. Therefore $(x_\mu(\nu, \lambda), 0)$ is a fixed point for Eq. (10). Since $[-M, M] \times \{0\}$ is compact, so we can find $-M \leq \mu_0, ..., \mu_k \leq M$ such that $[-M\,,\, M] \times \{0\} \subset \bigcup_{i=0}^{k} U_{\mu_i}$. Let $\mathcal{R}$ be an open rectangle with $[-M, M] \times \{0\} \subset \mathcal{R} \subset \bigcup_{i=0}^{k} U_{\mu_i}$. If $(\nu, \lambda) \in U_{\mu_i} \bigcup U_{\mu_j}$, $(i \neq j)$ then by the uniqueness of the solution for the implicit function theorem, $x_{\mu_i}(\nu, \lambda) = x_{\mu_j}(\nu, \lambda)$. This shows that we can define the C^1 map:

$$x : \mathcal{R} \to \mathbb{R}$$
$$(\mu, \lambda) \mapsto x_{\mu_i}(\mu, \lambda) \quad (\mu, \lambda) \in U_{\mu_i}.$$

The linear part of Eq. (12) at $(x(\mu, \lambda), 0)$ is

$$A(\mu, \lambda) = \begin{pmatrix} 0 & 1 \\ -\omega^2 + \mu f_x & \mu f_y \end{pmatrix}$$

where all partial derivatives are computed at $(x(\mu, \lambda) + \lambda, 0)$. The eigenvalues of $A(\mu, \lambda)$ are

$$\Gamma_{1,2} = \Gamma_{1,2}(\mu, \lambda) = \frac{\mu f_y \pm \sqrt{(\mu f_y)^2 + 4(\mu f_x - \omega^2)}}{2}. \tag{13}$$

If $(\mu f_y)^2 + 4(\mu f_x - \omega^2) < 0$ then the real part of $\Gamma_{1,2}$ is μf_y. By A_1, $Df(0, 0) = 0$ furthermore $(\mu f_y)^2 + 4(\mu f_x - \omega^2)$ varies continuously with respect to (μ, λ). Since for all $\mu \in [-M, M]$, $x(\mu, 0) = 0$ therefore we have

$$\lim_{\lambda \to 0} (\mu f_y)^2 + 4(\mu f_x - \omega^2) = -4\omega^2 < 0.$$

Since the closure of the set $\mathcal{R}$ is compact in $\mathbb{R}^2$ and $(\mu f_y)^2 + 4(\mu f_x - \omega^2)$ varies continuously with respect to (μ, λ), so $(\mu f_y)^2 + 4(\mu f_x - \omega^2)$ is uniformly continuous in $\mathcal{R}$. Therefore we can shrink $\mathcal{R}$ such that $[-M, M] \times \{0\} \subset \mathcal{R}$ and $(\mu f_y)^2 + 4(\mu f_x - \omega^2) < 0$. ∎

Lemma 2. *Let $(\mu, \lambda) \in \mathcal{R}$. Then $(\partial x / \partial \lambda)(\mu, \lambda) \neq -1$.*

Proof. We have

$$-\omega^2 x(\mu, \lambda) + \mu f(x(\mu, \lambda) + \lambda\,,\, 0) = 0 \quad (\mu, \lambda) \in \mathcal{R}.$$

Computing the derivative with respect to λ we get

$$-\omega^2 \frac{\partial x}{\partial \lambda}(\mu, \lambda) + \mu \frac{\partial f}{\partial x}(x(\mu, \lambda) + \lambda, 0)\Big(\frac{\partial x}{\partial \lambda}(\mu, \lambda) + 1\Big) = 0.$$

If $(\partial x / \partial \lambda)(\mu, \lambda) = -1$ then $-\omega^2 (\partial x / \partial \lambda)(\mu, \lambda) = 0$. But we assumed that $(\partial x / \partial \lambda)(\mu, \lambda) = -1$. This is a contradiction hence $(\partial x / \partial \lambda)(\mu, \lambda) \neq -1$. ∎

Theorem 4. *Let* $(\mu, 0) \in \mathcal{R}$*. If* $f_{xy}(0,0) \neq 0$ *then for* ε *sufficiently small Eq.* (9) *has periodic solutions.*

Proof. By Theorem 3, for each $(\mu, \lambda) \in \mathcal{R}$, $(x(\mu, \lambda), 0)$ is a fixed point for Eq. (12). Furthermore the real part of $\Gamma(\mu, \lambda)$ is $f_y(x(\mu, \lambda) + \lambda, 0)$. Let us define $g(\lambda) = f_y(x(\mu, \lambda) + \lambda, 0)$. Then $g(0) = 0$ and by Lemma 2

$$g'(0) = f_{xy}(0,0)\Big[\frac{\partial x}{\partial \lambda}(\mu, 0) + 1\Big] \neq 0.$$

This shows that for each $0 < |\lambda|$ sufficiently small, $g(\lambda) \neq 0$. Hence $(x(\mu, \lambda), 0)$ is a hyperbolic fixed point for Eq. (12). By the averaging theorem, for ε sufficiently small, Eq. (10) has hyperbolic periodic solutions therefore, by Lemma 1, Eq. (9) has periodic solutions. ■

4. Hopf Bifurcation and Invariant Sets

In this section we obtain some conditions for Eq. (10) in order to have a Hopf bifurcation. We consider the set $\mathcal{R}$ and the map $x : \mathcal{R} \to \mathbb{R}$ defined in Sec. 3.

Suppose that for $\lambda = 0$ a Hopf bifurcation occurs at the origin for Eq. (12), by a long but direct computation, we can see that the stability parameter of the Hopf bifurcation theorem is

$$\rho = \frac{\omega^2}{2} f_{xxy} + \frac{\omega^4}{2} f_{yyy} + \frac{1}{2} f_{xx} f_{xy} + \frac{\omega}{2} f_{xy} f_{yy} \tag{14}$$

where all partial derivatives are computed at $(0,0)$. If $\rho \neq 0$, then Eq. (12) has many hyperbolic periodic solutions. This enables us to find non-trivial bounded invariant sets for Eq. (9).

Theorem 5. *Consider Eq.* (9) *and suppose that the map* F *satisfies the conditions* A_1*,* A_2 *and* A_3*. Also assume that* $f_{xy}(0,0) \neq 0$ *and* $\rho \neq 0$ *where* ρ *is the stability parameter computed by* (14)*. Then for each* $(\mu, 0) \in \mathcal{R}$ *and* ε *sufficiently small Eq.* (9) *has non-trivial bounded invariant sets.*

Proof. By Theorem 3, for $(\mu, 0) \in \mathcal{R}$, the real part of $\Gamma(\mu, 0)$ is $\mu f_y(0,0) = 0$. Moreover by Lemma 2, $(\partial x)/(\partial \lambda)(\mu, 0) \neq -1$ therefore

$$\frac{d}{d\lambda} Rel\,\Gamma(\mu, 0) = \Big[\frac{\partial x}{\partial \lambda}(0,0) + 1\Big] f_{xy}(0,0) \neq 0.$$

By the Hopf bifurcation theorem, for $\lambda = 0$ a Hopf bifurcation occurs for Eq. (12) at the origin. Hence for $|\lambda|$ sufficiently small and $\lambda\mu\rho f_{xy} < 0$, Eq. (12) has hyperbolic periodic solutions. In Sec. 3 we saw that Eq. (12) is the time averaged equation of Eq. (10). Therefore, by averaging theorem, for ε sufficiently small, Eq. (10) has invariant torus. By Lemma 1 Eq. (9) has non-trivial bounded invariant sets. ■

Remark 2. Suppose that for $\lambda = \lambda_0 \neq 0$, a Hopf bifurcation occurs at $(x(\mu_0, \lambda_0), 0)$ for Eq. (12). By simple transfer we can assume that $\lambda_0 = 0$ and $x(\mu_0, \lambda_0) = 0$.

Hence by Theorem 4 and Theorem 5 Eq. (9) has periodic solutions and non-trivial bounded invariant sets.

Remark 3. Consider Eq. (12) and suppose that for $\lambda = 0$, a Hopf bifurcation occurs at the origin for it, also assume that $\rho = 0$. By Remark 1, Eq. (12) has many periodic solutions. Since these periodic solutions may be non-hyperbolic so they cannot imply the invariant torus for Eq. (10); however by Lemma 1 the equation $\dddot{x} + \omega^2 \dot{x} = \mu(f_x \dot{x} + f_{\dot{x}} \ddot{x})$ has many periodic solutions.

Corollary 1. *Let $e(t)$ be C^r, $r \geq 2$, and of periodic $2\pi\omega^{-1}$. Also let $f : \mathbb{R} \to \mathbb{R}$ be a C^{r+1} function such that $f(0) = Df(0) = 0$.*

(i) *If $\omega^3 D^3 f(0) + Df(0) \neq 0$ then for each $(\mu, 0) \in \mathcal{R}$ and ε sufficiently small the equation*

$$\dddot{x} + \omega^2 \dot{x} - \mu(\dot{x}^2 + x\ddot{x} + \ddot{x}Df(\dot{x}) + \varepsilon e'(t)) = 0$$

has non-trivial bounded invariant sets.

(ii) *If $D^2 f(0) \neq 0$ then for each $(\mu, 0) \in \mathcal{R}$ and ε sufficiently small the equation*

$$\dddot{x} + \omega^2 \dot{x} - \mu(\dot{x}^2 + x\ddot{x} + \dot{x}Df(x) + \varepsilon e'(t)) = 0$$

has non-trivial bounded invariant sets.

Proof.
(i) By Theorem 5 for $\lambda = 0$, a Hopf bifurcation occurs at the origin for the equation

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \mu((x + \lambda)y + f(y)) \end{cases}$$

By (14), we obtain the stability parameter $\rho = 2^{-1}\omega\left(\omega^3 D^3 f(0) + Df(0)\right) \neq 0$, hence for each $(\mu, 0) \in \mathcal{R}$ and ε sufficiently small the equation

$$\dddot{x} + \omega^2 \dot{x} - \mu(\dot{x}^2 + x\ddot{x} + \ddot{x}Df(\dot{x}) + \varepsilon e'(t)) = 0$$

has non-trivial bounded invariant sets.
(ii) By Theorem 5 for $\lambda = 0$, a Hopf bifurcation occurs at the origin for the equation

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \mu((x + \lambda)y + f(x)). \end{cases}$$

By (14), we obtain the stability parameter $\rho = D^2 f(0) \neq 0$, hence for each $(\mu, 0) \in \mathcal{R}$ and ε sufficiently small the equation

$$\dddot{x} + \omega^2 \dot{x} - \mu(\dot{x}^2 + x\ddot{x} + \dot{x}Df(x) + \varepsilon e'(t)) = 0$$

has non-trivial bounded invariant sets. ∎

5. Effect of the Quadratic Terms

In this section we study the effect of quadratic terms on Eq. (9). First, let us consider the equation

$$\dddot{x} + \omega^2 \dot{x} = \mu\big[a_{11}x^2 + a_{12}x\dot{x} + a_{13}x\ddot{x} + a_{22}\dot{x}^2 + a_{23}\dot{x}\ddot{x} + a_{33}\ddot{x}^2\big].$$

We can write the equation by

$$\frac{d}{dt}\Big[\ddot{x} + \omega^2 x - \mu\Big(\frac{a_{12}}{2}x^2 + a_{13}x\dot{x} + \frac{a_{23}}{2}\dot{x}^2\Big)\Big] = a_{11}x^2 + (a_{22} - a_{13})\dot{x}^2 + a_{33}\ddot{x}^2.$$

Now suppose that $x(t)$ is a nontrivial periodic solution for the equation with period T, then

$$\int_0^T \frac{d}{dt}\Big[\ddot{x} + \omega^2 x - \mu\Big(\frac{a_{12}}{2}x^2 + a_{13}x\dot{x} + \frac{a_{23}}{2}\dot{x}^2\Big)\Big] = \int_0^T a_{11}x^2 + (a_{22} - a_{13})\dot{x}^2 + a_{33}\ddot{x}^2.$$

Therefore

$$\underbrace{\Big[\ddot{x} + \omega^2 x - \mu\Big(\frac{a_{12}}{2}x^2 + a_{13}x\dot{x} + \frac{a_{23}}{2}\dot{x}^2\Big)\Big]_0^T}_{\text{since } x(t) \text{ is periodic, so it is equal to } 0} = \int_0^T a_{11}x^2 + (a_{22} - a_{13})\dot{x}^2 + a_{33}\ddot{x}^2$$

which implies

$$\int_0^T a_{11}x^2 + (a_{22} - a_{13})\dot{x}^2 + a_{33}\ddot{x}^2 = 0.$$

Suppose that a_{11}, $(a_{22} - a_{13})$, a_{33} are non-negative (non-positive) and at least one of them is non-zero. Since $x(t)$ is a non-trivial periodic solution so

$$\int_0^T a_{11}x^2 + (a_{22} - a_{13})\dot{x}^2 + a_{33}\ddot{x}^2 \neq 0.$$

This is a contradiction; hence the equation has no non-trivial periodic solution. Similar proof shows that Eq. (15) cannot has homoclinic orbits.

Corollary 2. *Consider Eq.* (15). *If* a_{11}, $(a_{22} - a_{13})$, a_{33} *are non-negative (non-positive) and at least one of them is non-zero then the equation has no non-trivial periodic solution and homoclinic orbit.*

Now we consider Eq. (15) with $a_{11} = (a_{22} - a_{13}) = a_{33} = 0$. Putting $2^{-1}a_{12} = a$, $a_{13} = 2b$, and $2^{-1}a_{23} = c$ we get

$$\frac{d}{dt}\big[\ddot{x} + \omega^2 x - \mu\left(ax^2 + 2bx\dot{x} + c\dot{x}^2\right)\big] = 0.$$

Reducing the equation to the second order equation we obtain

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \mu f(x + \lambda, y) \end{cases} \tag{16}$$

where $f(x,y) = ax^2 + 2bxy + cy^2$.

If $b \neq 0$ then $f_{xy}(0,0) = b \neq 0$, hence by Theorem 5, for $\lambda = 0$, a Hopf bifurcation occurs for Eq. (16) at the origin. Furthermore the stability parameter $\rho = 2b(a + \omega c)$. Hence if $a + \omega c \neq 0$ then for $\lambda\mu\rho f_{xy} < 0$ and $|\lambda|$ sufficiently small Eq. (16) has invariant torus.

By Lemma 1 and Theorems 4 and 5 we have

Corollary 3. *Suppose that $e(t)$ is C^r, $r \geq 2$ and of periodic $2\pi\omega^{-1}$ in t. If $b \neq 0$ then for ε sufficiently small the equation*

$$\dddot{x} + \omega^2\dot{x} = \mu(ax + b\dot{x})\dot{x} + \mu(bx + c\dot{x})\ddot{x} + \varepsilon e'(t)$$

has periodic solutions. Furthermore if $a + \omega c \neq 0$ then for ε sufficiently small the equation has non-trivial bounded invariant sets.

Now we consider Eq. (16) where $b = 0$

Case 1. We consider (16) with $a = b = 0$ and $c \neq 0$. In this case (16) has the following form

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \mu c y^2. \end{cases}$$

Proposition 1. *For each $\mu \in \mathcal{R}$, the equation*

$$\dddot{x} + \omega^2\dot{x} - \mu c(\dot{x}\ddot{x}) = 0 \quad (c \in \mathbb{R})$$

has many periodic solutions.

Proof. See [10]

Case 2. We consider Eq. (16) with $b = c = 0$ and $a \neq 0$. In this case Eq. (16) is

$$\begin{cases} \dot{x} = y \\ \dot{y} = a\mu\lambda^2 + (2a\mu\lambda - \omega^2)x + a\mu x^2. \end{cases}$$

Setting $\nu = a\mu$, we get

$$\begin{cases} \dot{x} = y \\ \dot{y} = \nu\lambda^2 + (2\nu\lambda - \omega^2)x + \nu x^2. \end{cases} \tag{17}$$

Proposition 2. *For each $\mu \in \mathcal{R}$ the equation*

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -\omega^2 x + a\mu x y \end{cases}$$

has many homoclinic orbits and periodic solutions. The homoclinic orbits make a 2-dimensional C^1 surface. In addition each homoclinic orbit lies on a C^1 2-dimensional orientable manifold and inside it is filled up with the periodic solutions.

Proof. See [10]

Now let us consider the equation

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \nu(x+\lambda)^2 + \varepsilon e(t). \end{cases} \tag{18}$$

where $e(t)$ is a C^2 and of period T. In Sec. 2, we saw that the time averaged equation of Eq. (18) is

$$\begin{cases} \dot{x} = y \\ \dot{v} = -\omega^2 x + \nu(x+\lambda)^2. \end{cases} \tag{19}$$

Eq. (19) has $(x(\nu,\lambda),0)$ as a hyperbolic saddle point where

$$x(\nu,\lambda) = \underbrace{\frac{\omega^2 - 2\nu\lambda}{2\nu}}_{\alpha} + \underbrace{\frac{\sqrt{\omega^4 - 4\lambda\nu\omega^2}}{2\nu}}_{\beta}.$$

hence for ε sufficiently small Eq. (18) has a hyperbolic periodic solution $\gamma(t) = (x(\nu,\lambda),0) + \mathcal{O}(\varepsilon)$; hence by Lemma 1 we have

Corollary 4. *Suppose that $e(t)$ is a C^2 periodic map. For each ε sufficiently small the equation*

$$\dot{\ddot{x}} + \omega^2 \dot{x} = \mu[x\dot{x} + \varepsilon e'(t)]$$

has periodic solutions.

Case 3. Let us consider Eq. (16) with $b = 0$ and $a, c \neq 0$. In this case (16) is reduced to

$$\begin{cases} \dot{x} = y \\ \dot{y} = a\mu\lambda^2 + (2a\mu\lambda - \omega^2)x + a\mu x^2 + c\mu y^2. \end{cases}$$

Setting $\nu = \mu a$ and $d = c/a$, we get

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \nu f(x+\lambda, y), \end{cases} \tag{20}$$

where $f(x,y) = x^2 + dy^2$. For each $\nu \neq 0$ and $(\nu,\lambda) \in \{(\nu,\lambda) : \nu\lambda < \omega^2/4\}$, Eq. (20) has two fixed points $(\alpha \pm \beta, 0)$ where α and β are as in case 2. $(\alpha+\beta\,,\,0)$ is a hyperbolic saddle point and the corresponding eigenvalues of $(\alpha-\beta\,,\,0)$ are purely imaginary. Putting $\bar{x} = x - (\alpha+\beta\,,\,0)$ and $\bar{y} = y$, we obtain

$$\begin{cases} \dot{\bar{x}} = \bar{y} \\ \dot{\bar{y}} = \sqrt{\omega^4 - 4\nu\lambda\omega^2}\,\bar{x} + \nu\bar{x}^2 + d\nu\bar{y}^2. \end{cases}$$

For convenience we drop the bars and obtain

$$\begin{cases} \dot{x} = y \\ \dot{y} = \sqrt{\omega^4 - 4\nu\lambda\omega^2}\,x + \nu x^2 + d\nu y^2. \end{cases} \tag{21}$$

This system has two fixed points $(0,0)$ and $(x_0,0) = (-\sqrt{\omega^4 - 4\nu\lambda\omega^2}/\nu, 0)$.

Proposition 3. *For each $\mu \in \mathcal{R}$ the system*

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -\omega^2 y + \mu(ax + cz)y \end{cases} \qquad (a, c \in \mathbb{R})$$

has many periodic solutions. In addition if $ac > 0$ then the system has many homoclinic orbits which make a C^1 2-dimensional surface. In this case each homoclinic orbit lies on a C^1 orientable 2-dimensional manifold and inside it is filled up with the periodic solutions.

Proof. See [10]

Suppose that $e(t)$ is a C^2 periodic map with period T. Consider the equation

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \nu(x + \lambda)^2 + d\nu y + \varepsilon e(t). \end{cases} \tag{22}$$

We saw that the time averaged equation is

$$\begin{cases} \dot{x} = \omega y \\ \dot{y} = \sqrt{\omega^4 - 4\nu\lambda\omega^2}x + \nu x^2 + d\nu y^2. \end{cases} \tag{23}$$

Eq. (23) has $(0, 0)$ as a hyperbolic fixed point hence for ε sufficiently small, Eq. (22) has a hyperbolic periodic solution $\gamma_\varepsilon(t) = (0, 0) + \mathcal{O}(\varepsilon)$. By Lemma 1 we have

Corollary 5. *Suppose that $e(t)$ is a C^2 periodic map. For ε sufficiently small the equation*

$$\dddot{x} + \omega^2 \dot{x} = \mu\big[ax\dot{x} + c\dot{x}\ddot{x} + \varepsilon e^{'}(t)\big]$$

has periodic solutions.

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