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A Note on Fractional Brownian Motion

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Abstract. We introduce a new approach to fractional Brownian motion by showing that it can be approximated in L^2 by semimartingales. Also, a relation between a fractional Brownian motion and a process of long memory is investigated.

1. Introduction

Many achievements have been made on fractional Brownian motion (fBm) in some recent years. Also, there are various approaches to fractional stochastic calculus by using some difficult tools as Malliavin Calculus (see, for example [1, 2]), theory of Wick product [3]. However, for many practical problems, one needs a simple method for study of stochastic dynamical systems driven by a fBm so that the method is not difficult for numerics.

In this note we prove that a fBm can be approximated in $L^2(\Omega)$ by semimartigales and show its relation to a time series of ARIMA type.

We recall that a fractional Brownian motion is a centered Gaussian process having covariance function $R(t,s)$ given by

$$R(s,t) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H}), \tag{1.1}$$

where the constant H is called the Hurst index, $0 < H < 1$. As we know, a fBm is not a semimartingale for $H \neq \frac{1}{2}$. For $H = \frac{1}{2}$, we have a standard Brownian motion. Put $\alpha = H - \frac{1}{2}$. It is known that a fractional Brownian motion W_t^H can be decomposed as follows:

$$W_t^H = \frac{1}{\Gamma(1+\alpha)}\Big[U_t + \int_0^t (t-s)^\alpha dW_s\Big], \tag{1.2}$$

where Γ is the gamma function,

$$U_t = \int_{-\infty}^{0} [(t-s)^\alpha - (-s)^\alpha] dW_s, \tag{1.3}$$

and W_t is a standard Brownian motion.

We suppose from now on that $0 < \alpha < \frac{1}{2}$ (or equivalently $\frac{1}{2} < H < 1$). The case where $0 < H < \frac{1}{2}$ has been studied in [5]. Then U_t is a process having absolutely continuous trajectories and we consider only the stochastic fractional integral

$$B_t = \int_0^t (t-s)^\alpha dW_s, \quad 0 < \alpha < \frac{1}{2}, \alpha = H - \frac{1}{2}. \tag{1.4}$$

The term B_t of the decomposition (1.2) of W_t^H plays an essential role in exhibiting a long range dependence and it is sometime defined as a fractional Brownian motion (see [1], [4]). B_t is not a semimartingale but it is approximated by semimartingales as shown below.

2. Approximation of B_t by Semimartingales

For every $\varepsilon > 0$ we define

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}} dW_s = \int_0^t (t-s+\varepsilon)^\alpha dW_s, \quad (0 < \alpha < \frac{1}{2} \text{ or } \frac{1}{2} < H < 1), \tag{2.1}$$

then we can prove that $(B_t^\varepsilon,\ t \geq 0)$ is a semimartingale. Indeed we have

Lemma. *The process* $(B_t^\varepsilon,\ t \geq 0)$ *has an Itô differential of the form:*

$$dB_t^\varepsilon = \big(\int_0^t \alpha (t-s+\varepsilon)^{\alpha-1} dW_s \big) dt + \varepsilon^\alpha dW_t. \tag{2.2}$$

Proof. We notice firstly that an application of Itô formula to the function $F(t, W_t) = \varphi(t).W_t$, where φ is a deterministic function will give us a formula of intergation-by-parts:

$$\int_0^t \varphi(s) dW_s = \varphi(t) W_t - \int_0^t W_s d\varphi(s). \tag{2.3}$$

Taking $\varphi(s) = (t-s+\varepsilon)^\alpha$ where t can be considered as a parameter we have $\varphi(t) = \varepsilon^\alpha$ and $d\varphi(s) = -\alpha(t-s+\varepsilon)^{\alpha-1} ds$ and then (2.3) becomes

$$\int_0^t (t-s+\varepsilon)^\alpha dW_s = \varepsilon^\alpha W_t + \alpha \int_0^t W_s (t-s+\varepsilon)^{\alpha-1} ds$$

or

$$B_t^\varepsilon = \varepsilon^\alpha W_t + \alpha d I(t), \tag{2.4}$$

where

$$I(t) = \int_0^t W_s (t-s+\varepsilon)^{\alpha-1} ds.$$

Now we have

$$dI(t) = I'(t)dt.$$

In taking account of the fact that the integrand of $I(t)$ and also the bound of $I(t)$ depend on the parameter t, we can apply the general formula of derivation for $I(t) = \int_{a(t)}^{b(t)} f(t,s)ds$:

$$I'(t) = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(t,s)ds + f[t,b(t)]b'(t) - f[t,a(t)]a'(t),$$

where

$$f(t,s) = W_s(t-s+\varepsilon)^{\alpha-1}, \quad a(t) = 0, \quad b(t) = t,$$
$$f(t,t) = W_t\varepsilon^{\alpha-1}, b'(t) = 1.$$

Therefore

$$\begin{aligned} I'(t) &= \int_0^t \frac{\partial}{\partial t} f(t,s)ds + f(t,t)b'(t) - f(t,0).0 \\ &= \int_0^t W_s \frac{\partial}{\partial t}(t-s+\varepsilon)^{\alpha-1}ds + \varepsilon^{\alpha-1}W_t. \end{aligned}$$

But

$$\frac{\partial}{\partial t}(t-s+\varepsilon)^{\alpha-1} = (\alpha-1)(t-s+\varepsilon)^{\alpha-2} = -\frac{\partial}{\partial s}(t-s+\varepsilon)^{\alpha-1}$$

then

$$\int_0^t W_s \frac{\partial}{\partial t}(t-s+\varepsilon)^{\alpha-1}ds = -\int_0^t W_s \frac{\partial}{\partial t}(t-s+\varepsilon)^{\alpha-1}ds.$$

An application of the formula of stochastic integration by parts will give us

$$-\int_0^t W_s \frac{\partial}{\partial t}(t-s+\varepsilon)^{\alpha-1}ds = -\varepsilon^{\alpha-1}W_t + \int_0^t (t-s+\varepsilon)^{\alpha-1}dW_s.$$

So we have

$$\begin{aligned} I'(t) &= -\varepsilon^{\alpha-1}W_t + \int_0^t (t-s+\varepsilon)^{\alpha-1}dW_s + \varepsilon^{\alpha-1}W_t \\ &= \int_0^t (t-s+\varepsilon)^{\alpha-1}dW_s. \end{aligned} \tag{2.5}$$

Hence

$$dI(t) = I'(t)dt = \Big[\int_0^t (t-s+\varepsilon)^{-\alpha-1}dW_s\Big]dt, \tag{2.6}$$

and then the formula (2.4) can be rewritten as

$$dB_t^\varepsilon = \varepsilon^\alpha dW_t + \alpha\Big[\int_0^t (t-s+\varepsilon)^{\alpha-1}dW_s\Big]dt$$

so B_t^ε is a semimartingale and the proof is thus complete.

Theorem 2.1. *B_t^ε converges to B_t in $L^2(\Omega)$ when ε tends to 0. This convergence is uniform with respect to $t \in [0, T]$.*

Proof. In a previous paper (see [5]) we have proved this theorem for the case where $0 < H < \frac{1}{2}$. Now we will extend this result for the case where $\frac{1}{2} < H < 1$.

Indeed we have

$$|(t-s+\varepsilon)^\alpha - (t-s)^\alpha| \le |\alpha|\varepsilon \sup_{0\le\theta\le1} |(t-s+\theta\varepsilon)^{\alpha-1}|$$

$$= |\alpha|\varepsilon(t-s)^{\alpha-1}, \quad \alpha = H - \frac{1}{2}. \tag{2.7}$$

$$E|B_t^\varepsilon - B_t|^2 = E\Big|\int_0^t \big[(t-s+\varepsilon)^\alpha - (t-s)^\alpha\big]dW_s\Big|^2$$

$$= \int_0^t \big|(t-s+\varepsilon)^\alpha - (t-s)^\alpha\big|^2 ds. \tag{2.8}$$

So if $\frac{1}{2} < H < 1$ that is $0 < \alpha < \frac{1}{2}$ we see from (2.7) that

$$\int_0^t \big|(t-s+\varepsilon)^\alpha - (t-s)^\alpha\big|^2 ds \le \alpha^2\varepsilon^2 \int_0^t |t-s|^{2\alpha-2} ds$$

$$= \alpha^2\varepsilon^2 \int_0^{t-\varepsilon} |t-s|^{2\alpha-2} ds + \alpha^2\varepsilon^2 \int_{t-\varepsilon}^t |t-s|^{2\alpha-2} ds$$

$$\le \alpha^2\varepsilon^2 \frac{\varepsilon^{2\alpha-1}}{1-2\alpha} + \alpha^2\varepsilon^2 \frac{\varepsilon^{2\alpha}}{1-2\alpha} = \frac{2\alpha^2}{1-2\alpha}\varepsilon^{2\alpha+1}. \tag{2.9}$$

Thus we have for every $t \in [0, T]$

$$\|B_t^\varepsilon - B_t\|^2 \le \frac{2\alpha^2}{1-2\alpha}\varepsilon^{2\alpha+1} \tag{2.10}$$

and

$$\sup_{0\le t\le T} \|B_t^\varepsilon - B_t\| \le C(\alpha)\varepsilon^{\alpha+\frac{1}{2}} \to 0, \tag{2.11}$$

where $C(\alpha) = \dfrac{\alpha\sqrt{2}}{\sqrt{1-2\alpha}}$ $(0 < \alpha < \dfrac{1}{2})$.

The inequality (2.11) shows that $B_t^\varepsilon \to B_t$ in $L^2(\Omega)$ uniformly with respect to $t \in [0, T]$.

3. Remark: Relation Between B_t and an ARIMA Process of Long Memory

Consider a time series of ARIMA type defined as

$$Y_s = (1-L)^{-d}\Phi(L)^{-1}\Theta(L)\varepsilon_s, \ s = 0, 1, 2, ..., [T] \tag{3.1}$$

where (ε_s) is a sequence of centered and uncorrelated random variables of the same variance σ, L is the lag operator, Φ and Θ are polynomials of L with roots outside of the unit disc. Suppose that the difference order d is greater than 1/2 so that Y is a non-stationary process. It is known that such an ARIMA process exhibits a long range dependence. It is a long memory process. And we will establish a relation between Y and the stochastic fractional integral B_t defined in (1.4).

Notice first that Y has a mobile average representation as follows

$$Y = \sum_{k=1}^{s} h_{s-k}^{(d)} \varepsilon_k \tag{3.2}$$

where mobile average coefficients h can be approximated as

$$h_s^{(d)} \approx \frac{\Theta(1)}{\Phi(1)\Gamma(1)} s^{d-1} \tag{3.3}$$

for large s, Γ being the gamma function.

Consider now a process of continuous time Z defined as

$$Z_r = \frac{1}{T^{d-\frac{1}{2}}} Y_{[Tr]},\ 0 \le r \le 1 \tag{3.4}$$

where $[x]$ stands for the integer part of x.

By some calculation and an application of Donsker theorem (see [4]) we get

$$\begin{aligned} Z_r &= \frac{1}{T^{d-\frac{1}{2}}} \sum_{k=1}^{[Tr]} h_{[Tr]-k}^{(d)} . \varepsilon_k \\ &\approx \frac{\sigma\Theta(1)}{\Phi(1)\Gamma(d)} \sum_{k=1}^{[Tr]} \Big(r - \frac{k}{T}\Big)^{d-1} \Big[W\Big(\frac{k}{T}\Big) - W\Big(\frac{k-1}{T}\Big)\Big] \end{aligned} \tag{3.5}$$

where W is a standard Brownian motion.

The sum in the right hand side of (3.5) is an integration sum corresponding to $\int_0^r (r-s)^{d-1} dW_s$, $0 \le r \le 1, d > \frac{1}{2}$. Then (3.5) can be rewritten as

$$Z_r \ \approx\ \frac{\sigma\Theta(1)}{\Phi(1)\Gamma(d)} \int_0^r (r-s)^{d-1} dW_s,\ 0 \le r \le 1,\ d > \frac{1}{2}. \tag{3.6}$$

Put $s = \frac{u}{T}$ and $\frac{r}{T} = t$ and notice that the Brownian motion W_s is self-similar, i.e. $W_s \equiv W_{\frac{u}{T}} \sim \frac{1}{T} W_u$ (identical in laws). Then we have

$$Z_r \equiv Z_{tT} = \frac{\sigma\Theta(1)}{\Phi(1)\Gamma(d)} . \frac{1}{T^d} \int_0^t (t-u)^{d-1} dW_u,\ 0 \le t \le T. \tag{3.7}$$

Put $d - 1 = -\alpha$, then $\alpha < \frac{1}{2}$ follows from $d > \frac{1}{2}$. We see from (3.7) that

$$\begin{aligned} Z_r &= C(\alpha) \int_0^t (t-u)^{-\alpha} dW_u, \\ &= C(\alpha) B_t,\ 0 \le t \le T, 0 < \alpha < \frac{1}{2}. \end{aligned} \tag{3.8}$$

where B_t is exactly the fractional stochastic integral defined in (1.4), and $C(\alpha)$ is some constant depending only on α.

So we see so far that the process B_t is in fact a limit case of an ARIMA process of long memory. That is why B_t exhibits also a long range dependence. And a stochastic dynamical system driven by the fractional noise B_t expresses some long term consequence system states, as frequently met in various applications to physics, telecommunication, finance, etc.

The approximation by semimartingales for B_t given in this note can be useful to supply a new approach to consider this kind of fractional stochastic dynamical system.

As an example we can consider a fractional model of Black-Scholes given by the equation

$$\begin{aligned} dS_t &= S_t(\mu dt + \nu dB_t^H), \\ S_t\big|_{t=0} &= S_0 \end{aligned} \tag{3.9}$$

where S_t is the price of the stock and μ and ν are constants and $B_t^H = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$.

The corresponding approximately fractional model of (3.9) is defined for each $\varepsilon > 0$ as follows

$$\begin{aligned} dS_t^\varepsilon &= S_t^\varepsilon(\mu dt + \nu dB_t^\varepsilon), 0 \le t \le T \\ S_t^\varepsilon\big|_{t=0} &= S_0, \text{ the same initial condition as in (3.9).} \end{aligned} \tag{3.10}$$

The solution of (3.10) is given by

$$S_t^\varepsilon = S_0 \exp\big(\frac{1}{2}\nu^2\varepsilon^{2\alpha}t + \nu\varepsilon^\alpha + \int_0^t H_s^\varepsilon ds\big). \tag{3.11}$$

Based on Theorem 2.1, one can show for the case $H > \frac{1}{2}$ that the solution S_t^ε converges in $L^2(\Omega)$ to a limit process S_t^* determined by

$$S_t^* = S_0 \exp\big(\mu t + \nu B_t\big) \tag{3.12}$$

as $\varepsilon \to 0$ and this convergence is uniform with respect to $t \in [0, T]$.

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