On Multicolored Forests in Complete Bipartite Graphs*

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Abstract. We present inequalities and majorization conditions on color distributions of the complete bipartite graph $K_{n,n+r}$ $(r \geq 0)$ which guarantee the existence of multicolored subgraphs: in particular, multicolored forests and trees.

Suppose that the edges of a graph G are colored using k colors where coloring is not necessarily proper. Let a_i be the number of edges of color i for $i = 1, 2, \ldots, k$. Call $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ the *color distribution* for this coloring. For an edge-coloring of a graph G, we say a subgraph H of G is *multicolored* provided no two of the edges of H are of the same color.

For a graph G with edge set E, let $e(G)$ denote the number of edges of G and $G[E_1]$ the spanning subgraph of G with edge set $E_1 \subseteq E$.

Let $\mathbf{p} = (p_1, p_2, \ldots, p_k)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_k)$ be two sequence of non-negative integers. Then we say $\mathbf{p}$ is *majorized* by $\mathbf{q}$, written $\mathbf{p} \preceq \mathbf{q}$, provided that when the subscripts are re-ordered so that $p_1 \leq p_2 \leq \ldots \leq p_k$ and $q_1 \leq q_2 \leq \ldots \leq q_k$, we have

$$p_1 + p_2 + \cdots + p_i \geq q_1 + q_2 + \cdots + q_i \quad \text{for} \quad i = 1, 2, \ldots, k,$$

with equality when $i = k$. It is easy to see that this definition for majorization is equivalent to the one in [4]. When we apply the majorization order to color distribution, sequences of different lengths are understood that zeros have been appended to the shorter sequence so that the two sequences may be compared.

In this note, we present inequalities and majorization conditions on color distributions of the complete bipartite graph $K_{n,n+r}$ ($r \geq 0$) which guarantee the existence of multicolored subgraphs: in particular, multicolored forests and trees. The main results are Theorems 1 and 2. They include the results for the case $r = 0$, which have recently been obtained by Brualdi and Hollingsworth [2].

We need the following three lemmas.

Lemma 1. (Gale and Ryser [4]) *Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be two vectors whose components are nonnegative integers. For $i = 1, 2, \ldots, m$, let $\delta_i = (1, 1, \ldots, 1, 0, 0, \ldots, 0)$ be a vector of n components with 1's in the first r_i positions and 0's elsewhere. Then there exists an $m \times n$ $(0, 1)$-matrix with row sum vector R and column sum vector S if and only if $S \preceq \bar{S}$ where $\bar{S}$ is the column vector of the matrix*

$$\bar{A} = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{bmatrix}.$$

Lemma 2. (Rado [3, 5]) *Let M be a matroid on a finite set E with rank function ρ and let $(A_1, A_2, \ldots, A_m)$ be a family of subsets of E. Then there exists an independent set $F = \{e_1, e_2, \ldots, e_m\}$ such that $e_i \in A_i$ for each $i = 1, 2, \ldots, m$ if and only if for every I with $\emptyset \neq I \subseteq \{1, 2, \ldots, m\}$,*

$$\rho(\cup_{i \in I} A_i) \geq |I|.$$

Note that, for a graph G with v vertices and edge set E and polygon matroid M, the rank function is $\rho(X) = v - t$ where t is the number of connected components of the spanning subgraph of G with edge set $X \subseteq E$.

Lemma 3. (Brualdi and Hollingsworth [2]) *Let G be a bipartite graph with p vertices and $t \geq p - k + 1$ connected components. Then*

$$e(G) \leq \lfloor k^2/4 \rfloor.$$

We are now ready to prove the main results.

Theorem 1. *If $G = K_{n,n+r}$ ($r \geq 0$) is colored with color distribution $(a_1, a_2, \ldots, a_p)$, then G admits a partition into multicolored subgraphs of sizes $2n - 1 + r, 2n - 3 + r, \ldots, 1 + r$ if and only if*

$$(a_1, a_2, \ldots, a_p) \preceq (1, 1, 2, 2, \ldots, n - 1, n - 1, n, n, \ldots, n) \qquad (1)$$

where the last $r + 1$ positions of the right sequence are n's. Moreover, if G is colored with color distribution $(a_1, a_2, \ldots, a_p)$ satisfying (1) where $0 \leq r < 1 + \sqrt{n}$, then it is possible to partition G into multicolored forests of sizes $2n - 1 + r, 2n - 3 + r, \ldots, 1 + r$.

Proof. Suppose that G is colored with color distribution $\mathbf{a} = (a_1, a_2, \ldots, a_p)$ and that G has been partitioned into multicolored subgraphs of sizes $2n - 1 + r, 2n - 3 + r, \ldots, 1 + r$. Consider the $n \times (2n - 1 + r)$ $(0, 1)$-matrix A with rows

indexed by subgraphs and columns by colors, in which the (i,j)-entry is 1 if and only if the ith subgraph has an edge of color j. Then the row sum vector is $R = (2n - 1 + r, 2n - 3 + r, \ldots, 1 + r)$, the column sum vector is $S = \mathbf{a}$, and the vector $\bar{S}$ in Lemma 1 is $(n, n, \ldots, n, n-1, n-1, \ldots, 2, 2, 1, 1)$ where the first $r + 1$ positions are n's. Hence G admits a partition into multicolored subgraphs of sizes $2n - 1 + r, 2n - 3 + r, \ldots, 1 + r$ if and only if there exists such a matrix A, and by Lemma 1 this is equivalent to (1).

Now assume that G is colored with color distribution $\mathbf{a}$ satisfying (1) where $0 \le r < 1 + \sqrt{n}$. We will show that G may be partitioned into multicolored forests of the desired sizes. For $i = 1, 2, \ldots, 2n - 1 + r$, let M_i denote the set of edges of color i. We establish the existence of edge-disjoint multicolored forests $F_1, F_2, \ldots, F_n$, where F_i contains $2i - 1 + r$ edges. We first show the existence of F_n. By Lemma 2, it suffices to show that in the polygon matroid of G, $\rho(U_I) \ge k$, where $U_I = \cup_{i \in I} M_i$ for each $k = 1, 2, \ldots, 2n - 1 + r$ and each $I \subseteq \{1, 2, \ldots, 2n - 1 + r\}$ with $|I| = k$.

Fix k and I, and let t be the number of connected components of $G_I = G[U_I]$. We need to show $2n + r - t \ge k$. Assume to the contrary that $t \ge 2n + 1 + r - k$. Since the edges of G_I comprise some of the M_i's, and $\mathbf{a}$ satisfies (1), we have $e(G_I) \ge a_1 + a_2 + \cdots + a_k$.

Case 1. $1 \le k \le 2n$. We have

$$
\begin{aligned}
e(G_I) &\ge a_1 + a_2 + \cdots + a_k \\
&\ge \begin{cases} 2(1 + 2 + \cdots + \frac{k}{2}) = \dfrac{k^2}{4} + \dfrac{k}{2} & \text{if } k \text{ is even} \\[2ex] 2(1 + 2 + \cdots + \frac{k-1}{2}) + \dfrac{k+1}{2} = \dfrac{k^2 - 1}{4} + \dfrac{k+1}{2} & \text{if } k \text{ is odd} \end{cases} \\[2ex]
&> \left\lfloor \dfrac{k^2}{4} \right\rfloor.
\end{aligned}
$$

Case 2. $2n + 1 \le k \le 2n - 1 + r$. Since $r < 1 + \sqrt{n}$, we have $2n - 2\sqrt{n} < k < 2n + 2\sqrt{n}$, and hence $n(1 + k - n) > k^2/4$. Therefore

$$
\begin{aligned}
e(G_I) &\ge a_1 + a_2 + \cdots + a_k \\
&\ge 2(1 + 2 + \cdots + n) + (k - 2n)n \\
&= n(1 + k - n) > \left\lfloor \dfrac{k^2}{4} \right\rfloor.
\end{aligned}
$$

Combining Cases 1 and 2, $e(G_I) > \lfloor k^2/4 \rfloor$. On the other hand, according to Lemma 3, $e(G_I) \le \lfloor k^2/4 \rfloor$, a contradiction.

Now consider $M'_1, M'_2, \ldots, M'_{2n-3+r}$ where $M'_i = M_i \backslash F_n$. (Note that $M_{2n-1+r} \backslash F_n$ and $M_{2n-2+r} \backslash F_n$ are empty since each of M_{2n-1+r} and M_{2n-2+r} consisted of only one edge). By applying Lemma 2 again to $U'_I = \cup_{i \in I} M'$ for $k = 1, 2, \ldots, 2n - r + 3$ and $I \subseteq \{1, 2, \ldots, 2n - 3 + r\}$ with $|I| = k$, we discover that there is a multicolored forest F_{n-1}. Continuing this way, we obtain all the multicolored forests. ∎

Theorem 2. *If* $(a_1, a_2, \ldots, a_{2n-1+r})$ *is a color distribution for* $G = K_{n,n+r}$ *$(r \geq 0)$ with subscripts ordered so that $a_1 \leq a_2 \leq \ldots \leq a_{2n-1+r}$, then every coloring of G with this distribution contains a multicolored spanning tree if and only if for every integer k with $k \leq 2n - 1 + r$,*

$$\sum_{i=1}^{k} a_i > k^2/4. \tag{2}$$

Proof. Note that $(a_1, a_2, \ldots, a_{2n-1+r})$ is a color distribution for $G = K_{n,n+r}$ with $a_1 \leq a_2 \leq \ldots \leq a_{2n-1+r}$.

Suppose first that for some integer k with $k \leq 2n - 1 + r$, $\sum_{i=1}^{k} a_i \leq k^2/4$. Then there are positive integers s, t with $\sum_{i=1}^{s+t} a_i \leq st$. Choose a biclique $B = K_{s,t}$ in G. B contains st edges, so it is possible to color its edges so that for $i = 1, 2, \ldots, s+t$, color i is applied to exactly a_i edges of B. Now, the rest of G maybe colored arbitrarily so that the color distribution is $(a_1, a_2, \ldots, a_{2n-1+r})$. Because there are only $2n - 1 + r$ colors, to form a multicolored spanning tree we must use every color. Hence we must choose the $s + t$ edges from B; but B has exactly $s + t$ vertices, we cannot avoid a cycle.

Now suppose that the color distribution $(a_1, a_2, \ldots, a_{2n-1+r})$ satisfies (2). For $i = 1, 2, \ldots, 2n - 1 + r$, let C_i be the collection of edges of color i. By Lemma 2, to show that there is a multicolored spanning tree in G. it suffices to show that in the polygon matroid of G, $\rho(U_I) \geq k$ where $U_I = \cup_{i \in I} C_i$ for each $k = 1, 2, \ldots, 2n - 1 + r$ and each $I = \{1, 2, \ldots, 2n - 1 + r\}$ with $|I| = k$.

Fix k and I, and let t be the number of connected components of $G_I = G[U_I]$. We need to show that $2n+r-t \geq k$. Assume to the contrary that $t \geq 2n+r-k+1$. Then from (2) we have $e(G_I) \geq a_1 + a_2 + \cdots + a_k > k^2/4$. On the other hand, according to Lemma 3, $e(G_I) \leq k^2/4$, a contradiction. ∎

References

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