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On the Denjoy–Perron–Henstock–Kurzweil Integral*

Dedicated to Prof. Dr. Nguyen Dinh Tri on the occasion of his 70th birthday

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Abstract. The present survey is an outline of some results on the completion of the Denjoy space due, in the one-dimensional case, to Ang, Lee and Vy [1], and, in the higher dimensional case, to Ang, Schmitt and Vy [2]. An application is given to an initial value problem with little smoothness on the initial value.

1. Introduction

The space of Henstock integrable functions on $[a, b]$, called the Denjoy space, has been studied by several authors (see [8] and the references therein). One disadvantage of the space is that it is not complete under the given norm

$$\|f\| = \sup\Big\{\Big|\int_a^x f(t)dt\Big| : a \le x \le b\Big\}.$$

The completion of the Denjoy space is, in fact, a closed subspace of the space of distributions on $[a, b]$ (cf. [1]). These one dimensional distributions are derivatives of continuous functions, they are said to be G-integrable, and their integral is, in fact, a generalization of the Denjoy–Perron–Henstock integral.

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We first make a definition. We say that a distribution f is G-integrable on $[a, b]$ if there is a continuous function F on $[a, b]$ whose (distributional) derivative is f and then the G-integral of f on $[a, b]$ is given by

$$(G)\int_a^b f = F(b) - F(a).$$

The following results have been proved in [1]:

(i) If f, g are G-integrable on $[a, b]$ and α, β are real numbers, then $\alpha f + \beta g$ is also G-integrable on $[a, b]$ and

$$(G)\int_a^b (\alpha f + \beta g) = \alpha (G)\int_a^b f + \beta (G)\int_a^b g.$$

(ii) If f is G-integrable on $[a, b]$ and $[c, d] \subset [a, b]$, then f is G-integrable on $[c, d]$, i.e., f is a distribution defined on $D[c, d]$ and the derivative in the distribution sense of a continuous function.

(iii) If f is G-integrable on $[a, c]$ and on $[c, b]$ and if f is G-integrable on $[a, b]$, then

$$(G)\int_a^b f = (G)\int_a^c f + (G)\int_c^b f.$$

The proofs of the foregoing facts can be found in [1] where some convergence theorems are proved for sequences of (one-dimensional) G-integrable distributions.

The remainder of the paper deals with multidimensional forms of the Denjoy-Perron-Henstock-Kurzweil integral. A number of multidimensional integration theories, that are extensions of, or have relations to, the Denjoy–Perron–Henstock –Kurzweil integral, were studied in recent years in e.g. [4], [8] (generalized Riemann integral), [3] (generalized Denjoy integral), [9] (GP-integral), [7] (BV-integral) and others. Compared with those integrals, the one presented here has several advantages. First, one main goal of the above theories is to weaken the smoothness conditions on the vector fields in the Divergence theorem. In the quoted works, the continuous differentiability of the vector fields was replaced by their continuity and their pointwise, or asymptotic or a.e. differentiability (with some other supplementary conditions). Here, we shall remove all hypotheses about differentiability and give a Divergence theorem for the class of all continuous vector fields (in fact, for a larger class consisting of distributions). In our case, the derivatives are no longer functions but are distributions. Second, as was remarked in [8], unlike the one-dimensional case, one drawback of the known multidimensional integrals is that one cannot develop in the same system both Divergence and Fubini type theorems. This can be done, however, in the integration theory presented here. The third point concerns convergence theorems. In some of the previous integration theories, the convergence theorems are rather complicated (see, e.g., Sec. 21, Chap. 5, 2b) and in some others, they seem to be incomplete (for example, in [9] or [3], there were forms of monotone convergence theorems, but dominated convergence type theorems were missing). In our paper [2], we proved some simple general convergence theorems that admit both monotone and dominated convergence theorems as direct consequences.

To simplify the presentation, we study here integration theory in the plane. The general case can be carried out in much the same way. The remainder of the paper consists of six sections. In Sec. 2, we define the class of G-integrable distributions, and an integration theory for it. Elementary properties such as linear operations, relation with Lebesgue integration ... are considered in this section. In Sec. 3, we consider some Fubini type theorems for G-integrable distributions. A Green's theorem is proved in Sec. 4. Sec. 5 is devoted to convergence theorems. The final Sec. 6 deals with an application to differential equations.

2. Definition of the Class $G(Q)$ and Integration on $G(Q)$

Let $a, b, c, d \in \mathbb{R}$, $a < b$ and $c < d$. In the sequel, we usually denote by Q the (open) rectangle $(a, b) \times (c, d)$ in $\mathbb{R}^2$. For simplicity, we put $\partial = \partial_{12} = \partial_{21}$ in $D'(Q)$ where $D(Q)$ is the space of test functions and consider the class

$$G(Q) = \{\partial F \in D'(Q) : F \in C(\overline{Q})\}.$$

We shall study $G(Q)$ and define an integration on it. We need the following

Lemma 1. *Let $F \in C(\overline{Q})$. Then $\partial F = 0$ (in $D'(Q)$) if and only if there exist $H \in C([a, b])$, $K \in C([c, d])$ such that*

$$F(x, y) = H(x) + K(y) \quad \forall x, y \in \overline{Q}. \tag{1}$$

For $f \in G(Q)$, we put

$$\mathcal{F}(f) = \{F \in C(\overline{Q}) : \partial F = f \quad \text{in} \quad D'(Q)\}.$$

Then, we have

Lemma 2. *Let $f \in G(Q)$, $F_1, F_2 \in \mathcal{F}(f)$. Then*

$$\begin{aligned} &F_1(x, y) + F_1(a, c) - F_1(a, y) - F_1(x, c) \\ &= F_2(x, y) + F_2(a, c) - F_2(a, y) - F_2(x, c) \end{aligned} \tag{2}$$

for all $(x, y) \in \overline{Q}$. Moreover, there exists a unique $F(f) \in \mathcal{F}(f)$ such that

$$F(f)(a, y) = F(f)(x, c) = 0 \quad \forall x \in [a, b],\ y \in [c, d]. \tag{3}$$

In view of the above lemmas, we can set the following

Definition 1. *Let $f \in G(Q)$ (f is said to be G-integrable on Q) and let*

$$Q' = (a', b') \times (c', d') \subset Q.$$

We put

$$\int\limits_{Q'} f = F(f)(b',d') + F(f)(a',c') - F(f)(a',d') - F(f)(b',c') \tag{4}$$

where $F(f)$ is given by Lemma 2.

From Lemma 2, it is seen that (4) still holds if we replace $F(f)$ by any $F \in \mathcal{F}(f)$. When $Q' = Q$, (4) becomes

$$\int\limits_{Q} f = F(f)(b,d). \tag{5}$$

Now, we put

$$\widehat{C}(Q) = \big\{ f \in C(\overline{Q}) : f(a,y) = f(x,c) = 0 \quad \forall x \in [a,b], y \in [c,d] \big\}$$

and for $f \in G(Q)$

$$\|f\| = \sup \Big\{ \Big| \int\limits_{(a,x)\times(c,y)} f \Big| : (x,y) \in \overline{Q} \Big\}.$$

It can be verified that $\widehat{C}(Q)$ is a closed (and thus a Banach) subspace of $C(\overline{Q})$ (with the usual norm $\|f\|_\infty = \max\limits_{(x,y)\in\overline{Q}} |f(x,y)|$) and that $\|.\|$ is a norm on $G(Q)$.

In fact we have

Theorem 1. $\big(G(Q), \|.\|\big)$ *is a separable Banach space which is isomorphic to* $\big(\widehat{C}(Q), \|.\|_\infty\big)$.

The next theorem shows that the integral on $G(Q)$ is an extension of the Lebesgue integral (in what follows we use $(L)\int$ to denote the Lebesgue integral).

Theorem 2. *If we identify $f \in L^1(Q)$ with the distribution*

$$f : \phi \mapsto (L) \int\limits_Q f\phi, \quad \phi \in D(Q),$$

then $f \in G(Q)$ and $\int\limits_Q f = (L) \int\limits_Q f$. Moreover, $C(\overline{Q})$ is dense in $\big(G(Q), \|.\|\big)$ and $G(Q)$ is the completion of $C(\overline{Q})$ $\big($or $L^1(Q)\big)$ with respect to the norm

$$\|f\| = \sup \Big\{ \Big| (L) \int\limits_a^x \int\limits_c^y f \Big| : (x,y) \in \overline{Q} \Big\}.$$

3. Fubini Theorems for G-Integrable Distributions

In this section, we consider some Fubini type theorems for the G-integral. These theorems will be applied to some initial value problems for the two-dimensional wave equation with nonsmooth initial data.

3.1. We first make some remarks on traces of integrals of G-integrable distributions.

Let $f \in C(\overline{Q})$ and $x \in [a, b]$. Consider the function

$$\int_a^x f(\xi, \cdot)d\xi : [c, d] \to \mathbb{R}$$

$$y \mapsto \int_a^x f(\xi, y)d\xi.$$

For $y \in [c, d]$, we have

$$\int_a^x f(\xi, y)d\xi = \frac{d}{dy}\int_c^y \int_a^x f(\xi, \eta)d\xi d\eta = \frac{d}{dy}[F(f)(x, y)].$$

Thus

$$\int_a^x f(\xi, \cdot)d\xi = [F(f)(x, \cdot)]' \text{ on } [c, d].$$

Generalizing to the case $f \in G(Q)$, we have the following

Definition 2. *Let $f \in G(Q)$, $x \in [a, b]$, $y \in [c, d]$. We define*

$$\int_a^x f(\xi, \cdot)d\xi = \big[F(f)(x, \cdot)\big]' \text{ in } D'(c, d),$$

$$\int_c^y f(\cdot, \eta)d\eta = \big[F(f)(\cdot, y)\big]' \text{ in } D'(a, b).$$

Since

$$F(f)(x, \cdot) \in C([a, b]) \text{ and } F(f)(\cdot, y) \in C([c, d]),$$

we have (cf. [1])

$$\int_a^\infty f(\xi, \cdot)d\xi \in G(c, d) \quad \text{and} \quad \int_c^y f(\cdot, \eta)d\eta \in G(a, b).$$

By [1], we can integrate these distributions over $[c, d]$ and $[a, b]$ respectively. Note that there is consistency in the above definition.

Following is a Fubini type theorem.

Theorem 3. *For all $f \in G(Q)$, we have*

$$\int_Q f = \int_a^b \Big(\int_c^d f(\cdot, \eta)d\eta\Big) = \int_c^d \Big(\int_a^b f(\xi, \cdot)d\xi\Big).$$

For a proof, we remark first that the above repeated integrals exist in the sense of [1]. Since $F(f)(\cdot, d)(a) = F(f)(a, d) = 0$, one has (see [1])

$$F(f)(\cdot,d) = F\Big(\int_c^d f(\cdot,\eta)d\eta\Big)$$

and

$$\int_a^b \Big(\int_c^d f(\cdot,\eta)d\eta\Big) = F(f)(b,d) = \int_Q f.$$

We obtain the first equality. The second is proved similarly.

3.2. In this section, we derive another form of Fubini's theorem for a subclass of $G(Q)$. To this end, we define

$$\begin{aligned} G_1^*(Q) = \big\{\partial_1 F \quad &(\text{distributionally}) \ : F \in L^1(Q),\\ &F(\cdot,y) \in C([a,b]) \ \text{ for a.e. } \ y \in [c,d],\\ &\text{and there exists } \ g = g(F) \in L^1(c,d)\\ &\text{such that } |F(x,\cdot)| \le g \quad \forall x \in [a,b]\big\},\\ G_2^*(Q) = \big\{\partial_2 F \ \ &(\text{distributionally}) : F \in L^1(Q),\\ &F(x,\cdot) \in C([c,d]) \quad \text{for a.e. } \ x \in [a,b],\\ &\text{and there exists } g = g(F) \in L^1(a,b)\\ &\text{such that } |F(\cdot,y)| \le g \quad \forall y \in [c,d]\big\}. \end{aligned}$$

Then we have

Theorem 4. *If $f \in G(Q) \cap G_1^*(Q)$, then the function*

$$y \mapsto \int_a^b f(\cdot,y), \quad y \in [c,d]$$

is Lebesgue integrable on $[c,d]$ and

$$\int_Q f = \int_c^d \Big(\int_a^b f(\cdot,y)dy\Big).$$

Hence for all $f \in G(Q) \cap G_1^*(Q) \cap G_2^*(Q)$, we have

$$\int_Q f = \int_c^d \Big(\int_a^b f(\cdot,y)dy\Big) = \int_a^b \Big(\int_c^d f(x,\cdot)dx\Big).$$

4. Green's Theorem for G-Integrable Distributions

In this section, the boundary $\Gamma = \partial Q = \{a,b\} \times [c,d] \cup [a,b] \times \{c,d\}$ is oriented in the usual (counter-clockwise) direction. Let $pdx + qdy$ be differential form in Q, where $p, q \in D'(Q)$ is a (distributional) vector field. If the traces of p and q on the sides of Q can be defined and if the integrals

$$\int_{[a,b]\times\{c\}} p\big|_{[a,b]\times\{c\}} = \int_a^b p(\cdot,c) \quad \int_{[a,b]\times\{d\}} p\big|_{[a,b]\times\{d\}} = \int_a^b p(\cdot,d)$$

$$\int_{\{a\}\times[c,d]} q\big|_{\{a\}\times[c,d]} = \int_c^d q(a,\cdot) \quad \int_{\{b\}\times[c,d]} q\big|_{\{b\}\times[c,d]} = \int_c^d q(b,\cdot)$$

exist in some sense, then we can define the integral of the form $pdx + qdy$ in the usual way by

$$\int_\Gamma pdx + qdy = \int_a^b p(\cdot,c) - \int_a^b p(\cdot,d) + \int_c^d q(b,\cdot) - \int_c^d q(a,\cdot).$$

We have the following form of Green' s theorem for $G(Q)$.

Theorem 5. *Suppose the vector field $(p,q) \in G_1(Q) \times G_2(Q)$ where*

$$G_i(Q) = \{\partial_i F : F \in C(\overline{Q})\}, \quad i = 1,2.$$

Then

(i) *the traces $p(\cdot,c), p(\cdot,d)$ (resp. $q(a,\cdot), q(b,\cdot)$) are (one-dimensional) G-integrable distributions on $[a,b]$ (resp. $[c,d]$);*

(ii) *$\partial_1 q, \partial_2 p \in G(Q)$ and we have Green's formula*

$$\int_\Gamma pdx + qdy = \int_Q (\partial_1 q - \partial_2 p).$$

5. Convergence Theorems

We examine conditions on sequences $\{f_n\} \subset G(Q)$ in order that the convergence of f_n to f (in some sense) with $f \in G(Q)$ implies that $\int_Q f_n \to \int_Q f$. We recall that a sequence $\{F_n\}$ on Q is said to be locally uniformly bounded in Q if for each $x \in Q$ there exists a neighborhood $U_x \subset Q$ of x such that $\sup\{|F_n(y)| : y \in U_x,\ n \in \mathbb{N}\} < \infty$.

We have the following convergence theorem

Theorem 6. *Let $\{f_n\}$ be a sequence in $G(Q)$ such that*

(i) *The sequence of primitives $\{F(f_n)\}$ is locally uniformly bounded in Q.*

(ii) *$\{F(f_n)\}$ converges pointwise on $\overline{Q}$ to a continuous function on $\overline{Q}$.*

Then $\{F(f_n)\}$ converges distributionally to a G-integrable distribution f and moreover $\int_Q f_n \to \int_Q f$ as $n \to \infty$.

This theorem admits the following variant

Theorem 7. *Let $\{f_n\}$ be a sequence in $G(Q)$ such that (i) and (ii) above hold and that $f_n \to f$ in $D'(Q)$. Then $f \in G(Q)$ and $\int\limits_Q f_n \to \int\limits_Q f$ as $n \to \infty$.*

From these theorems, some familiar consequences can be derived

Corollary 1. *Let $\{f_n\}$ be a sequence in $G(Q)$ such that $f_n \to f$ in $D'(Q)$ and that $\{F(f_n)\}$ is equicontinuous on $\overline{Q}$. Then $f_n \to f$ in $G(Q)$ and $\int\limits_Q f_n \to \int\limits_Q f$ as $n \to \infty$.*

Corollary 2. (Monotone convergence theorem for G-integral) *Let $\{f_n\}$ be a sequence in $G(Q)$ such that $f_1 \le f_2 \le \cdots \le f_n \le \cdots$ and that $\int\limits_Q f_n \to a$ as $n \to \infty$. Then $f_n \to f$ in $G(Q)$ and $\int\limits_Q f = a$.*

Corollary 3. (Dominated convergence theorem for G-integral) *Let $\{f_n\}$ be a sequence in $G(Q)$ such that $f_n \to f$ in $D'(Q)$. Suppose there exist $g, h \in G(Q)$ satisfying $g \le f_n \le h$, $\forall n \in \mathbb{N}$. Then $f \in G(Q)$ and $\lim_{n\to\infty} \int\limits_Q f_n = \int\limits_Q f$.*

6. An Application to Differential Equations

An application to G-integration is given to an elementary "initial value" problem for the wave equation, where an initial value data that must not necessarily be smooth and solutions are sought in the class $G(Q)$.

Specifically, the following simple problem is considered in the unit square $Q = (0,1)^2$

$$\begin{cases} u_{xy} = \partial_{12} u = f & \text{in } Q \\ u(x,x) = h(x) & \\ u_y(x,x) = \partial_2 u(x,x) = g(x) & \text{for } x \in (0,1). \end{cases} \tag{IVP}$$

It is assumed that g and h are continuous on $[0,1]$ and that $f \in G_1(Q)$ where

$$G_1(Q) = \{\partial_1 F : F \in C(\overline{Q})\}.$$

The problem is to find solutions u of (IVP) in the class

$$A = \{u \in C(\overline{Q}) : \partial_2 u \in C(\overline{Q})\}.$$

As shown in [2], the problem admits a unique solution given by

$$u(x,y) = \int_0^x \int_0^y f + h(x) - \int_0^x \int_0^x f - \int_0^x \Big[g(\xi) - \int_0^\xi f(t,\xi)dt\Big] d\xi \\ + \int_0^x g - \int_0^x \int_0^\xi f(t,\xi)dt d\xi.$$

It is noted that wave equations with nonsmooth (distributional) data have been studied extensively since the appearance of distributions (cf. [5] and the references therein, see also [14] for a nice and elementary presentation). In departure from the classical approach, in which the initial conditions are included as source terms, our approach here is, in some sense, between the classical and distributional ones, and we can relax smoothness conditions on the source terms and the problem can now be written as an integral equation.

In closing, we quote the recent book on the subject with a different point of view by Salomon Leader, *"The Kurzweil–Henstock integral and its differentials on* $\mathbb{R}$ *and* $\mathbb{R}^2$*"*, Marcel Dekker, Basel 2001.

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