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Another Classification of the Class of Quasi-Games Which Become Fairer with Time*

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Abstract. Let G be the class of all nondecreasing functions from N to N. We shall define a partial order on G and classify the class of all quasi-games fairer with time into a nondecreasing family of subclasses, directed by $g \in G$ for which the smallest element coincides with the set of all games fairer with time.

1. Notations and Preliminaries

In this paper we are dealing with a complete probability space $(\Omega, \mathcal{A}, P)$, N the set of all positive integers and $(\mathcal{A}_n)$ an increasing sequence of complete sub-σ-fields of $\mathcal{A}$ with $\mathcal{A}_n \uparrow \mathcal{A}$. By T we denote the set of all bounded stopping times with respect to $(\mathcal{A}_n)$. Then equipped with the usual order "$\leq$", given by: $\sigma \leq \tau$ if and only if $\sigma(\omega) \leq \tau(\omega)$, a.s, T becomes a directed set and N can be regarded as a cofinal subset of T.

To avoid any confusion, for a given subset Γ of T, $p \in N$ and $\tau \in T$ with $p \leq \tau$ we shall use the following additional notations

$$\Gamma(p) = \{\gamma \in \Gamma \,|\, \gamma \geq p\},$$
$$\Gamma(p, \tau) = \{\gamma \in \Gamma \,|\, p \leq \gamma \leq \tau\}.$$

Given a sub-σ-field $\mathbb{B}$ of $\mathcal{A}$, we denote by $L^0(\mathbb{B})$ the set of all (equivalence classes of) $\mathbb{B}$-measurable random variables and $L^1(\mathbb{B})$ the Banach space of all elements $X \in L^0(\mathbb{B})$ with

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$$E\big(|X|\big) = \int_\Omega |X| dP < \infty.$$

From now on, we shall consider only sequences (X_n) in $L^0(\mathcal{A})$ which are adapted to $(\mathcal{A}_n)$, i.e., each $X_n \in L^0(\mathcal{A}_n)$.

For such a sequence (X_n) and $\tau \in T$, we define the random variable X_τ and the subset $\mathcal{A}_\tau$ of $\mathcal{A}$ by

$$X_\tau(\omega) = X_{\tau(\omega)}(\omega),\ \omega \in \Omega,$$

and

$$\mathcal{A}_\tau = \{A \in \mathcal{A} \mid A \cap \{\tau = k\} \in \mathcal{A}_k,\ k \in N\}.$$

Then it is known (cf. [7]) that $(\mathcal{A}_\tau,\ \tau \in T)$ is an increasing family of complete sub-σ-fields of $\mathcal{A}$ and each X_τ is $\mathcal{A}_\tau$-measurable. In addition, if (X_n) is integrable then so is $(X_\tau,\ \tau \in T)$.

For other related notions, we refer to [2, 7]. In this paper we start with the following definition.

Definition 1.1. *A sequence (X_n) in $L^1(\mathcal{A})$ is said to be a game which becomes fairer with time if for every $\epsilon > 0$ there exists $p \in N$ such that, for all $n \in N(p)$ we have*

$$\sup_{p \le q \le n} P\big(\big|E^q(X_n) - X_q\big| > \epsilon\big) < \epsilon, \tag{1}$$

where for a given $X \in L^1(\mathcal{A})$ and $\tau \in T$ we denote by $E^\tau(X)$ the conditional expectation of X, given $\mathcal{A}_\tau$.

Games fairer with time were introduced by Blake (1970) [1] who proved that: *every real-valued game (X_n) which becomes fairer with time, uniformly bounded a.s. by an integrable function, converges in L^1.*

Some years later Mucci [6] and Subramanian [8] extended (independently) this result to the uniformly integrable case. Further, applying the structure results of Talagrand [9], Luu [3] extended the Blake's conclusion to obtain stochastic convergence of the following martingale-like sequences, more general than games fairer with time.

Definition 1.2. *A sequence (X_n) is called a quasi-game fairer with time, briefly a quasi-game* (cf. [4]) *if for every $\epsilon > 0$ there exists p such that for every $m > p$ there is $p_m \ge m$ such that for all $n \ge p_m$ we have*

$$\sup_{p \le q \le m} P\Big(\big|E^q(X_n) - X_q\big| > \epsilon\Big) < \epsilon. \tag{2}$$

It is clear that by definition, *every game fairer with time is a quasi-game.* Using Example 2.3 [4] and Theorem 2.2 [5], the reader could construct many quasi-games which are not games fairer with time. To know more how large the class of all quasi-games is, we shall prove in the next section, as the main result of this paper that the class of all quasi-games can be classified into an increasing directed family of subclasses for which the smallest element coincides with the class of all games fairer with time.

2. Main Results

In order to present the main results of the paper, let us consider the set G of all functions "almost nondecreasing" from N to N, i.e., $g \in G$ if and only if there exists $n_0(g) \in N$ such that g is nondecreasing on the set $\{n \geq n_0(g)\}$. Further, let define on G the partial order "$'\leq$", given by

$$f \,'< g \quad \Leftrightarrow \begin{cases} \text{card } \{f > g\} < \infty \\ \text{card } \{g > f\} = \infty, \end{cases}$$

and

$$f \overset{\prime}{=} g \quad \Leftrightarrow \quad \text{card } \{f \neq g\} < \infty.$$

It is easily checked that endowed with the partial order, G becomes a directed set. Moreover, we get the following simple characterization of the class of all quasi-games in term of $(G, '\leq)$.

Lemma 2.1. *A sequence (X_n) is a quasi-game if and only if there exists some $g \in G$ such that (X_n) is a game of size g, i.e., for every $\epsilon > 0$ there exists p such that for every $m > p$ and $n \geq m + g(m)$, the requirement* (2) *is satisfied.*

In particular, (X_n) is a game which becomes fairer with time if and only if it is a game of size 1.

Proof. By a modification of the proof of Lemma 2.1 [5], one can obtain the result.

However, as the lemma is the first important starting point for the next main classification, we prefer to give its another direct structural proof for the sake of completeness.

Indeed, if (X_n) is a quasi-game fairer with time, then by definition, we can construct a strictly increasing sequence $\big(q(n)\big)$ of N such that for every k, there exists an increasing sequence $\big(q_n(k)\big)$ such that for every $m > q(k)$ and $n \geq q_m(k)$ we have

$$\sup_{q(k) \leq s \leq m} P\Big(\big|E^s(X_n) - X_s\big| > \frac{1}{k}\Big) < \frac{1}{k}. \tag{3}$$

Now we define the function g from N to N as follows. For $m \leq q(1)$ we set $g(m) = 1$ but for $m > q(1)$, i.e., $q(p) < m \leq q(p+1)$, for some $p \geq 1$, we take

$$g(m) = \sum_{s=1}^{p} q_m(s). \tag{4}$$

It is clear that $g \in G$.

We shall prove that (X_n) is a game of size g. To do this, let $\epsilon > 0$ be given. Then $1/k < \epsilon$ for some k. Thus if $m > q(k)$ and $n \geq m + g(m)$ we have

$$q(k + j - 1) < m \leq q(k + j), \text{ for some } j \geq 1,$$

and then by (4) we have

$$n \geq m + g(m) = m + \sum_{s=1}^{k+j-1} q_m(s) \geq q_m(k).$$

Therefore, (3) implies that

$$\sup_{q(k)\le s\le m} P\Big(\big|E^s(X_n)-X_s\big|>\epsilon\Big)<\epsilon.$$

It means that (X_n) is a game of size g which proves the "only if"-part of the lemma.

Conversely, suppose that (X_n) is a game of size g, for some $g\in G$. Set $p_m=m+g(m),\ m\in N$. It is not hard to check that (p_m) satisfies Definition 1.2, where p can be always assumed to be larger than $n_0(g)$. Therefore, (X_n) is a quasi-game. It completes the proof of the first part of the lemma.

To prove the second part of the lemma, we assume first that (X_n) is a game which becomes fairer with time. Then for every $\epsilon>0$, there exists p such that for all $n>p$, (1) is satisfied.

For ϵ and p as above, let $m>p$ and $n\ge m+1$. Then (1) implies (2). It means that (X_n) is a game of size 1.

Conversely, suppose that (X_n) is a game of size 1. Then for every $\epsilon>0$, there exists $p\in N$ such that for any $m>p$ and $n\ge m+1$, (2) is satisfied.

Now let $n\in N$ with $n>p$. Choose $m=n-1$. Then $n\ge m+1$. Thus by (2) we have

$$\sup_{p\le q\le n} P\Big(\big|E^q(X_n)-X_q\big|>\epsilon\Big)=\sup_{p\le q\le n-1} P\Big(\big|E^q(X_n)-X_q\big|>\epsilon\Big)<\epsilon.$$

It means that (X_n) is a game which becomes fairer with time. This completes the proof. ∎

Having the above characterization we can prove the following classification.

Theorem 2.2. *When g runs over the directed set $(G,{}'\!\le)$, the set of all quasi-games can be classified into a nondecreasing family of the subclasses of games of size g for which the smallest element is just the class of games which becomes fairer with time. Moreover, if $f,g\in G$ with $f\,{}'\!< g$ then the class of all games of size f is strictly contained in the class of games of size g.*

Proof. The first claim of the theorem follows immediately from Lemma 2.1. To prove the next main part of the theorem, we note first that if $h\in G$ then the sequence $\big(a_m(h)\big)$ will "almost strictly increase", i.e., there exists $k_0(h)\in N$ such that $a_{(\cdot)}(h)$ is a function strictly increasing on the set $\{m\ge k_0(h)\}$, where $a_m(h)=m+h(m),\ m\in N$.

This implies that

$$b_n(h)=m \quad \text{if and only if} \quad a_m(h)\le n<a_{m+1}(h), \tag{5}$$

where $b_n(h)=\max\{m\in N\,|\,a_m(h)\le n\},\ n\ge a_1(h)$.

In particular, we have

$$b_{a_m(h)}(h)=m \quad \text{for} \quad m\ge k_0(h), \tag{6}$$

and

$$b_n(h)+h\big[b_n(h)\big]\le n. \tag{7}$$

Now let $f, g \in G$ satisfying the condition $f \,'\!< g$. We shall prove the main part of the theorem by constructing a game of size g which is not a game of size f.

Indeed, let $(\Omega, \mathcal{A}, P)$ be the usual Lebesgue probability space on $[0,1)$. For $n \in N$, let $a_n = 2^n.n!$ and denote by Q_n the partition of $[0,1)$ in a_n intervals of equal length. Then $a_n = 2n.a_{n-1},\ n \geq 2$. Set $\mathcal{A}_n = \sigma - (Q_n)$. Since $g \in G$, there exists $n_0(g) \in N$ such that g is nondecreasing on the set $\{n \geq n_0(g)\}$. Let k denote the first index $m \in N$ such that

$$m \geq \max\{n_0(g),\ n_0(f),\ k_0(g),\ n_0\} \quad \text{and} \quad g(m) > f(m),$$

where $n_0 = 1 + \max\{m \mid f(m) > g(m)\} < \infty$. This implies that

$$f(s) \leq g(s) \quad \text{for every} \quad s \geq k,$$

and

$$g(k) \geq f(k) + 1 \geq 2. \tag{8}$$

On the other hand, since $b_{(\cdot)}(g)$ is a nondecreasing function on the set $\{n \geq a_1(g)\}$, by (5) one obtains

$$b_n(g) \geq b_{a_k(g)}(g) = k, \quad n \geq a_k(g).$$

This together with (8) implies

$$g\big(b_n(g)\big) \geq g(k) \geq 2.$$

Hence, by (7) for $n \geq a_k(g)$ the following holds

$$n \geq b_n(g) + g\big[b_n(g)\big] \geq b_n(g) + 2. \tag{9}$$

Now, we can define the sequence (X_n) as follows:

For $n < a_k(g)$, set $X_n = 0$.

For any but fixed $n \geq a_k(g)$, by (9) we prefer to define X_n in the following both cases:

a) If $n = b_n(g)+2$ we set $X_n = -n$ or $X_n = n$, resp., on the first two intervals or on the last two intervals, resp., of $\mathcal{A}_n$ which are contained in the $(2p-1)$-th or in the $(2p)$-th interval of $\mathcal{A}_{b_n(g)+1}$, resp., with $1 \leq p \leq \frac{a_{b_n(g)+1}}{2}$.

b) If $n > b_n(g) + 2$ we set

$$X_n = -\big[b_n(g) + 2\big] \quad \text{or} \quad X_n = b_n(g) + 2, \ \text{resp.},$$

on the first $2^{n-(b_n(g)+1)}n(n-1)\ldots(b_n(g)+3)$ intervals or on the last $2^{n-(b_n(g)+1)}$ $n(n-1)\ldots(b_n(g)+3)$ intervals, resp., of $\mathcal{A}_n$ which are contained in the $(2p-1)$-th or $(2p)$-th interval of $\mathcal{A}_{b_n(g)+1}$, resp., with $1 \leq p \leq \frac{a_{b_n(g)+1}}{2}$.

We shall show that the sequence (X_n) constructed in such a way will satisfy all the aforementioned requirements.

Indeed, we claim first that for $n \geq a_k(g)$ we have

$$E^{b_n(g)+1}(X_n) = -1 \quad \text{or} \quad E^{b_n(g)+1}(X_n) = 1, \tag{10}$$

resp., on the $(2p-1)$-th or on the $(2p)$-th interval of $\mathcal{A}_{b_n(g)+1}$, resp., with $1 \leq p \leq \frac{a_{b_n(g)+1}}{2}$.

To see the claim, we note that for $n > m$ the following holds

$$a_n = 2^{(n-m)}.n(n-1)\cdots(m+1)a_m.$$

Thus for $n \geq a_k(g)$ and $m = b_n(g)+1$, we have

$$a_n = 2^{n-(b_n(g)+1)}.n(n-1)\cdots(b_n(g)+2).a_{b_n(g)+1}. \tag{11}$$

Now to see (10) in the first case a) when $n = b_n(g)+2$, it is enough to note that on the interval I_{2p-1} of $\mathcal{A}_{b_n(g)+1}$ with $1 \leq p \leq \frac{a_{b_n(g)+1}}{2}$, we have

$$\int_{I_{2p-1}} X_n dP = -\frac{2n}{a_n} = -\frac{2n}{2na_{b_n(g)+1}} = \frac{-1}{a_{b_n(g)+1}}.$$

Then $E^{b_n(g)+1}(X_n) = -1$ on the interval I_{2p-1} of $\mathcal{A}_{b_n(g)+1}$ with any

$$1 \leq p \leq \frac{a_{b_n(g)+1}}{2}.$$

Similarly, on the interval I_{2p} of $\mathcal{A}_{b_n(g)+1}$ with $1 \leq p \leq \frac{a_{b_n(g)+1}}{2}$, we get

$$E^{b_n(g)+1}(X_n) = 1.$$

Next, for the second case b) when $n > b_n(g)+2$, by taking an even interval I_{2p} of $\mathcal{A}_{b_n(g)+1}$ with $1 \leq p \leq \frac{a_{b_n(g)+1}}{2}$, we have

$$\int_{I_{2p}} X_n dP = \frac{2^{n-(b_n(g)+1)}.n(n-1)\cdots(b_n(g)+3).(b_n(g)+2)}{a_n} =$$

$$= \frac{2^{n-(b_n(g)+1)}.n(n-1)\cdots(b_n(g)+3).(b_n(g)+2)}{2^{n-(b_n(g)+1)}.n(n-1)\cdots(b_n(g)+3).(b_n(g)+2).a_{b_n(g)+1}} = \frac{1}{a_{b_n(g)+1}}.$$

This shows that

$$E^{b_n(g)+1}(X_n) = 1,$$

on the interval I_{2p} of $\mathcal{A}_{b_n(g)+1}$ with $1 \leq p \leq \frac{a_{b_n(g)+1}}{2}$.

Similarly, on the interval I_{2p-1} of $\mathcal{A}_{b_n(g)+1}$ with $1 \leq p \leq \frac{a_{b_n(g)+1}}{2}$ we have

$$E^{b_n(g)+1}(X_n) = -1.$$

This proves the claim (10) which implies that for $q < b_n(g)+1$ we have

$$E^q(X_n) = 0. \tag{12}$$

To complete the construction, we prove first that (X_n) is a game of size g. For this purpose, let $p, m, n \in N$ with $m > p$ and $n \geq m + g(m)$, we conclude that $m < b_n(g)+1$. Indeed, suppose on the contrary that $m \geq b_n(g)+1$. Then $m > b_n(g) = \max\{s \mid s + g(s) \leq n\}$. Hence $m + g(m) > n$. This contradicts the hypothesis on m and n and the conclusion is verified. Thus by (12) we get

$$E^m(X_n) = 0.$$

Then, for every $q \leq m$, we obtain

$$\left|E^q(X_n) - X_q\right| = \left|X_q\right|.$$

Therefore to prove that (X_n) is a game of size g, it suffices to show that (X_n) goes to zero in probability. To check this, let $\epsilon > 0$ be given. Clearly

$$\{|X_q| > \epsilon\} \subset \{X_q \neq 0\}.$$

It is then enough to show

$$P\big(X_q \neq 0\big) \overset{q\to\infty}{\longrightarrow} 0.$$

Indeed, let $n \geq a_k(g)$. We have to consider the following two cases:

If $n = b_n(g) + 2$, then by the definition of (X_n) we have

$$P\big(X_n \neq 0\big) = \frac{2}{a_n}.a_{b_n(g)+1} = \frac{2a_{b_n(g)+1}}{2na_{b_n(g)+1}} = \frac{1}{n} = \frac{1}{b_n(g)+2}.$$

Further, if $n > b_n(g) + 2$, then

$$P\big(X_n \neq 0\big) = \frac{1}{a_n}.2^{n-(b_n(g)+1)}.n(n-1)\cdots(b_n(g)+3).a_{b_n(g)+1}$$
$$= \frac{2^{n-(b_n(g)+1)}.n(n+1)\cdots(b_n(g)+3).a_{b_n(g)+1}}{2^{n-(b_n(g)+1)}.n(n+1)\cdots(b_n(g)+3).(b_n(g)+2).a_{b_n(g)+1}} = \frac{1}{a_{b_n(g)+2}}.$$

Thus, in general we have

$$P\Big(\big|X_q\big| > \epsilon\Big) \leq P\big(X_q \neq 0\big) = \frac{1}{b_q(g)+2} \overset{q\to\infty}{\longrightarrow} 0.$$

This implies that (X_n) is a game of size g.

Now, to show the second required property of (X_n), we set

$$U(f,g) = \{v \geq a_k(g) \,|\, f(v) < g(v)\}.$$

The hypothesis "$f \,'\!< g$" implies

$$\operatorname{card} U(f,g) = \infty.$$

On the other hand, by the definition of $a_n(h)$ with $h \in G$, we have also

$$U(f,g) = \{v \geq a_k(g) \,|\, a_v(f) < a_v(g)\}.$$

But by (6):

$$b_{a_v(g)}(g) = v = b_{a_v(f)}(f), \quad v \in U(f,g).$$

This together with (5) yields

$$s = b_{a_v(f)}(g) < b_{a_v(g)}(g) = v, \quad v \in U(f,g), \tag{13}$$

since $a_v(f) < a_v(g),\ v \in U(f,g)$.

Finally, let $p > a_k(g)$ and $v \in U(f,g)$ with

$$p \leq b_{a_v(f)}(g) + 1.$$

Take $m = v,\ n = a_v(f)$ and $q(v) = b_{a_v(f)}(g) + 1$. Then by (13)

$$p \leq q(v) \leq v$$

and

$$P\Big(\big|E^{q(v)}(X_n) - X_{q(v)}\big| = 1\Big) \geq P\Big(\big\{\big|E^{q(v)}(X_n)\big| = 1\big\} \cap \big\{X_{q(v)} = 0\big\}\Big).$$

But by Equation (10) we have

$$P\Big(\big\{\big|E^{q(v)}(X_n)\big| = 1\big\}\Big) = 1.$$

Then

$$P\Big(\big|E^{q(v)}(X_n) - X_{q(v)}\big| = 1\Big) \geq P\Big(\big\{X_{q(v)} = 0\big\}\Big) = \\ = 1 - P\big(X_{q(v)} \neq 0\big) = 1 - \frac{1}{b_{q(v)}(g) + 2}.$$

Thus letting v run over $U(f, g)$, the above estimations show that (X_n) cannot be a game of size f, and the theorem is proved. ■

Here, as the referee suggested, we would like to compare this new classification with the previous one, given in [5] by Luu. For this purpose, let denote as in [5] by F the set of all functions from N to N. Then as noted in [5], F becomes a directed set with the other partial order “ ${}^*{\leq}$” given by

$$f \ {}^*{<}\ g \quad \Leftrightarrow \quad \text{card}\ \{g \leq f\} < \infty,$$

and

$$f \ \overset{*}{=}\ g \quad \Leftrightarrow \quad \text{card}\ \{f \neq g\} < \infty.$$

Moreover, one can verify also the following.

Lemma 2.3. *Endowed with the order “${}'{\leq}$”, F becomes a directed set.*

We remark that if $f \ {}^*{\leq}\ g$, taken in the sense of Luu [5] then $f \ {}'{\leq}\ g$. Conversely, the following example shows that the order “${}'{\leq}$” is much finer than “${}^*{\leq}$”. Indeed, let us take $f, g \in F$ as follows:

$$f(m) = 1, \quad m \in N,$$

and

$$g(m) = \begin{cases} 1, & \text{if } m \text{ is an even number,} \\ 2, & \text{if } m \text{ is an odd number.} \end{cases}$$

Then it is easily seen that $f \ {}'{<}\ g$ but f and g cannot be compared each with the other in “${}^*{\leq}$”. Moreover, the following theorem explains why Luu [5] could classify the class of all quasi-games fairer with time into an increasing family of subclasses indexed only by $g \in (F, {}^*{\leq})$ but not by $f \in (F, {}'{\leq})$.

Theorem 2.4.
a) *There exist functions $f, g \in F$ with $f \ {}'{<}\ g$ but $\mathcal{G}^f = \mathcal{G}^g$, where given $h \in F$, $\mathcal{G}^h$ denotes the class of all games (X_n) of size h.*

b) *The set* $\{b_n(f)\,|\,n \geq a_1(f)\}$ *is just* N *if and only if* f *is nondecreasing, where given* $h \in F$*, we denote:*

$$a_m(h) = m + h(m), \;\; m \in N$$

and

$$b_n(h) = \max\{m \in N\,|\,a_m(h) \leq n\}\,, \; n \geq a_1(h). \tag{14}$$

Proof.

a) First, we show that there exist two functions $f, g \in F$ with $f \,'\!< g$ but $b_n(f) = b_n(g), \; n \geq 3$.

Indeed, we take two functions $f, g \in F$ defined as follows:

$$f(m) = \mathrm{mod}_2(m) + 1, \quad m \in N,$$

where $\mathrm{mod}_2(2p-1) = 1, \; \mathrm{mod}_2(2p) = 0, \; p \in N$ and

$$g(m) = \begin{cases} f(m), & \text{if ; } \mathrm{mod}_2(m) = 0 \\ f(m) + 1, & \text{if } \mathrm{mod}_2(m) \neq 0. \end{cases}$$

It is easily seen that $f \,'\!< g$. We shall prove that for all $n \geq 3$ we have $b_n(f) = b_n(g)$. Indeed, by the same definition we see that

$$f(m) = \begin{cases} 1, & \text{if } m \text{ is even} \\ 2, & \text{if } m \text{ is odd,} \end{cases}$$

and

$$g(m) = \begin{cases} 1, & \text{if } m \text{ is even} \\ 3, & \text{if } m \text{ is odd.} \end{cases}$$

Hence, by a direct computation of (14) we have

$$\begin{aligned} b_{2k+1}(f) &= \max\{m\,|\,m + f(m) \leq 2k+1\} \\ &= \max\{2k-1, \, 2k\} = 2k, \; k \geq 1. \\ b_{2k}(f) &= \max\{m\,|\,m + f(m) \leq 2k\} \\ &= \max\{2k-3, \, 2k-2\} = 2k-2, \; k \geq 2. \end{aligned} \tag{15}$$

Similarly, we also have

$$\begin{aligned} b_{2k+1}(g) &= \max\{m\,|\,m + g(m) \leq 2k+1\} \\ &= \max\{2k-3, \, 2k\} = 2k, \; k \geq 1 \end{aligned}$$

$$\begin{aligned} b_{2k}(g) &= \max\{m\,|\,m + g(m) \leq 2k\} \\ &= \max\{2k-3, \, 2k-2\} = 2k-2, \; k \geq 2. \end{aligned} \tag{16}$$

By (15) and (16) for all $n \geq 3$ we obtain

$$b_n(f) = b_n(g). \tag{17}$$

Further, we prove that (17) implies $\mathcal{G}^f = \mathcal{G}^g$. Indeed, suppose that $(X_n) \in \mathcal{G}^f$. By definition, for every $\epsilon > 0$ there exists $p \geq 2$ such that for every $m > p$ and $n \geq m + f(m)$ we have

$$\sup_{p \leq q \leq m} P\Big(\big|E^q(X_n) - X_q\big| > \epsilon\Big) < \epsilon. \tag{18}$$

To prove that $(X_n) \in \mathcal{G}^g$, let $k, n \in N$ with $k > p$ and $n \geq k + g(k)$. We shall prove that

$$\sup_{p \leq q \leq k} P\Big(\big|E^q(X_n) - X_q\big| > \epsilon\Big) < \epsilon. \tag{19}$$

To see (19), we take $m = b_n(g)$. Then $m \geq k > p$. On the other hand, by (17), we have

$$n \geq b_n(f) + f\big[b_n(f)\big] = m + f(m).$$

Hence, by (18) we obtain (19). Consequently

$$\mathcal{G}^f \subset \mathcal{G}^g.$$

But it is worth noting that the symmetric property (17) would be applied also to g. Then symmetrically we have

$$\mathcal{G}^g \subset \mathcal{G}^f.$$

Thus

$$\mathcal{G}^f = \mathcal{G}^g.$$

It completes the proof of a).
b) We start with the proof of the necessity condition. Indeed, from the definition of $b_n(f)$ it follows that $b_{(\cdot)}(f)$ is a nondecreasing function and

$$b_n(f) + f\big[b_n(f)\big] \leq n, \ n \geq a_1(f). \tag{20}$$

Now, let $v \in N$. Set

$$k_v = \max\{n \,|\, b_n(f) = v\}.$$

To prove that f is nondecreasing, we show first the following inequality

$$k_{v-1} < v + f(v) \leq k_v, \ v \geq 2. \tag{21}$$

To see this, let $v \in N$. By the assumption that $N = \{b_n(f) \,|\, n \geq a_1(f)\}$, there exists q such that $b_q(f) = v$. Therefore by the definition of k_v we have $q \leq k_v$. But by (20) we have

$$v + f(v) = b_q(f) + f\big[b_q(f)\big] \leq q.$$

Hence

$$v + f(v) \leq q \leq k_v.$$

It means that the second inequality of (21) holds.

Next, to prove the first inequality of (21), we assume the contrary, that for some $v \geq 2$ we have

$$k_{v-1} \geq v + f(v).$$

Then by the nondecreasing property of $b_{(\cdot)}(f)$ it follows that

$$b_{k_{v-1}}(f) \geq b_{v+f(v)}(f). \tag{22}$$

But by definition we have

$$b_{k_{v-1}}(f) = v - 1.$$

Then (22) becomes

$$v - 1 \geq b_{v+f(v)}(f). \tag{23}$$

On the other hand, by the same definition we have also

$$b_{v+f(v)}(f) \geq v.$$

This with (23) implies

$$v \leq b_{v+f(v)}(f) \leq v - 1.$$

It is impossible. Thus the double inequality (21) is completely proved.

Now, applying the inequality (21) we have

$$(v-1) + f(v-1) \leq k_{v-1} < v + f(v), \ v \geq 2.$$

Then

$$f(v-1) < f(v) + 1.$$

Equivalently

$$f(v-1) \leq f(v), \quad v \geq 2.$$

In other words, f is nondecreasing. It completes the proof of the necessary condition.

To prove the sufficient condition, let f be a nondecreasing function. We have to show

$$\big\{b_n(f) \,|\, n \geq a_1(f)\big\} = N.$$

Indeed, the first inclusion

$$\{b_n(f) \,|\, n \geq a_1(f)\} \subset N, \tag{24}$$

is natural. Then it remains to check only the converse inclusion. To do this, let $v \in N$. We claim that

$$b_n(f) = v \text{ if and only if } v + f(v) \leq n < (v+1) + f(v+1).$$

To see the "only if"-part of the claim, suppose that $b_n(f) = v$ for some $n \in N$. Then, by definition $v + f(v) \leq n$. To prove the second property of n, we suppose by the contrary that

$$n \geq (v+1) + f(v+1).$$

Then by the same definition of $b_n(f)$ we have

$$b_n(f) \geq v + 1 > v.$$

It is impossible. Thus

$$n < (v+1) + f(v+1).$$

It proves the "only if"-part of the claim.

Conversely, suppose that for some $n, v \in N$ we have

$$v + f(v) \le n < (v+1) + f(v+1).$$

Then, by the definition of $b_n(f)$, the left inequality guarantees that

$$b_n(f) \ge v.$$

Suppose that $b_n(f) > v$. It follows that

$$b_n(f) \ge v + 1.$$

On the other hand, we always have

$$n \ge b_n(f) + f\big[b_n(f)\big].$$

Then by the nondecreasing property of $b_{(\cdot)}(f)$ we get

$$n \ge (v+1) + f(v+1).$$

This contradicts the assumption on n. Thus

$$b_n(f) = v.$$

It proves the claim which implies

$$N \subset \{b_n(f) \,|\, n \ge a_1(f)\}.$$

Therefore by (24), this part and so the theorem is proved. ∎

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