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On Continuity of General Exhaustive Set Functions

Dedicated to Professor Nguyen Duy Tien on the occassion of his 60th birthday

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Abstract. It is shown that any continuous from below exhaustive set function given on a σ-ring and with values in a regular topological space is continuous at $\emptyset$.

1. Introduction

The exhaustive additive set functions with values in Banach spaces, topological groups and semigroups play an important role (see [2–4, 8]). We define this notion for a set function with values in an arbitrary topological space and relate it with other continuity properties in this general setting. In particular, the statement formulated in the abstract setting is proved. By using this result in [1] it is shown that in a certain class of topological abelian monoids X the continuity at $\emptyset$ of any X-valued additive set function already implies its (sequential) continuity everywhere. All necessary definitions are given in Sec. 2.

2. Definitions

2.1. Convergence Structure in $\mathcal{P}(\Omega)$

We shall use ordinary set-theoretic notations and terminology.

Let (Ω) be a non-empty set. The collection $\mathcal{P}(\Omega)$ of all subsets of Ω will be equipped with usual set-theoretic operations and order $\subset$.

Let $(A_j)_{j\in J}$ be a non-empty family of subsets of Ω. Clearly, the set $\cup_{j\in J} A_j$ coincides with the least upper bound of $(A_j)_{j\in J}$ in the partially ordered set

$(\mathcal{P}(\Omega), \subset)$ and $\cap_{j\in J} A_j$ coincides with the greatest lower bound of $(A_j)_{j\in J}$ in the partially ordered set $(\mathcal{P}(\Omega), \subset)$.

For a sequence (A_n) of sets, as in case of the sequences of real numbers, we can define

$$\limsup_n A_n := \inf_n \sup_{k\geq n} A_k = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

and

$$\liminf_n A_n := \sup_n \inf_{k\geq n} A_k = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$

If (A_n) is a sequence of sets and A is a set, then we shall say that (A_n) *converges* to A and write $\lim_n A_n = A$ or $A_n \to A\,(n\to\infty)$ if

$$\limsup_n A_n = A = \liminf_n A_n.$$

It is easy to observe that for every sequence (A_n) of sets we have

$$\liminf_n A_n \subset \limsup_n A_n.$$

From this we can conclude that for a sequence (A_n) of sets we have $\lim_n A_n = \emptyset$ if and only if $\limsup_n A_n = \emptyset$. Note also that

$$\lim_n A_n = \emptyset,\;\; B_n \subset A_n,\;\; n = 1, 2, \dots \Longrightarrow \lim_n B_n = \emptyset.$$

A sequence (A_n) of sets is said to be *increasing* (resp., *decreasing*) if $A_n \subset A_{n+1}$, $n = 1, 2, \dots$ (resp., $A_n \supset A_{n+1}$, $n = 1, 2, \dots$).

Notice that if the sequence (A_n) is increasing (resp., decreasing), then $\lim_n A_n = A$ if and only if $\cup_{k=1}^{\infty} A_k = A$ (resp., $\cap_{k=1}^{\infty} A_k = A$).

Similar notions can be defined also for nets of sets, but we shall use only the sequences.

The symmetric difference between sets A_1 and A_2 will be denoted by $A_1 \smile A_2$.[1]

Lemma 2.1. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets, A be a set and $(k_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers. Then*
(a) $\limsup_n A_n = A \Longrightarrow \limsup_n A_{k_n} \subset A$.
(b) $\liminf_n A_n = A \Longrightarrow \liminf_n A_{k_n} \supset A$.
(c) $\lim_n A_n = A \Longrightarrow \lim_n A_{k_n} = A$.
(d) $\lim_n A_n = A \Longleftrightarrow \lim_n (A_n \smile A) = \emptyset$.

A straightforward proof of this lemma is left to the reader.

A family of sets $(A_i)_{i\in I}$ with $\mathrm{card}\,(I) \geq 2$ will be called *disjoint* if $A_i \cap A_j = \emptyset$ for all $i, j \in I$ with $i \neq j$.

[1]The triplet $(\mathcal{P}(\Omega), \smile, \cap)$ is an abelian ring in the algebraic sense of this word; i.e., the pair $(\mathcal{P}(\Omega), \smile)$ is abelian group with the neutral element $\emptyset$, the pair $(\mathcal{P}(\Omega), \cap)$ is an abelian monoid (with the neutral element Ω) and the distributive law holds.

Lemma 2.2. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets.*

(a) *If $(A_n)_{n\in\mathbb{N}}$ is a disjoint sequence of sets, then* $\lim_n A_n = \emptyset$.

(b) *If $D_n := A_n \setminus \cup_{j=n+1}^{\infty} A_j$, $n = 1, 2, \dots$, then $(D_n)_{n\in\mathbb{N}}$ is a disjoint sequence of sets.*

For a given $A \subset \Omega$ we denote 1_A the indicator function of the set A.
Note that if $A_1, A_2 \subset \Omega$, then $1_{(A_1 \smile A_2)}(\omega) = |1_{A_1}(\omega) - 1_{A_2}(\omega)|$, $\forall \omega \in \Omega$.

Lemma 2.3. *Suppose that A_n, $n = 1, 2, \dots$ and A are subsets of a fixed set Ω, then we have*

$$\limsup_n A_n = A \iff \limsup_n 1_{A_n}(\omega) = 1_A(\omega) \quad \forall \omega \in \Omega.$$
$$\liminf_n A_n = A \iff \liminf_n 1_{A_n}(\omega) = 1_A(\omega) \quad \forall \omega \in \Omega.$$
$$\lim_n A_n = A \iff \lim_n 1_{A_n}(\omega) = 1_A(\omega) \quad \forall \omega \in \Omega.$$

The last observation implies that the considered convergence in $\mathcal{P}(\Omega)$ is topological. Namely, $\mathcal{P}(\Omega)$ can be identified through the map $A \to 1_A$ with set $\{0, 1\}^{\Omega}$ of all functions from Ω to $\{0, 1\}$, which carries the natural pointwise convergence topology (which is a metrizable topology if and only if the set Ω is at most countable). If we 'transport' this topology in $\mathcal{P}(\Omega)$, then the convergence of a sequence of sets in this topology will coincide with our convergence.[2]

2.2. Algebra, σ-Algebra and Their Companions

We recall now several notions from measure theory (cf. [6]).

Fix a non-empty $\mathfrak{A} \subset \mathcal{P}(\Omega)$.

(1) $\mathfrak{A}$ is called a *ring* if for every $A, B \in \mathfrak{A}$ the sets $A \cup B$, $A \setminus B$ also are members of $\mathfrak{A}$.[3]

$\mathfrak{A}$ is called a *σ-ring* if $\mathfrak{A}$ is a ring such that for every infinite sequence $A_1, A_2, \dots, A_n, \dots \in \mathfrak{A}$, the set $\cup_{n=1}^{\infty} A_n$ also is a member of $\mathfrak{A}$.

(2) $\mathfrak{A}$ is called a *(Boolean) algebra* or a *'field'* if for every $A, B \in \mathfrak{A}$ the sets $A \cap B$ and $A^c := \Omega \setminus A$ also are members of $\mathfrak{A}$.

It is easy to see that $\mathfrak{A}$ is an algebra iff $\mathfrak{A}$ is a ring such that $\Omega \in \mathfrak{A}$.

$\mathfrak{A}$ is called a *(Boolean) σ-algebra* if $\mathfrak{A}$ is a (Boolean) algebra such that for every infinite sequence $A_1, A_2, \dots, A_n, \dots \in \mathfrak{A}$, the set $\cup_{n=1}^{\infty} A_n$ is also a member of $\mathfrak{A}$.

Observe that $\mathfrak{A}$ is σ-algebra iff for every $A \in \mathfrak{A}$ its complement A^c is a member of $\mathfrak{A}$ and for every infinite sequence $A_1, A_2, \dots, A_n, \dots \in \mathfrak{A}$, the set $\cup_{n=1}^{\infty} A_n$ also is a member of $\mathfrak{A}$.

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[2]It turns out that the triplet $(\mathcal{P}(\Omega), \smile, \cap)$ equipped with 'transported' from $\{0, 1\}^{\Omega}$ pointwise convergence topology presents an example of a compact Hausdorff topological ring. This observation is useful when Ω is countable.

[3]It is easy to see that a non-empty $\mathfrak{A} \subset \mathcal{P}(\Omega)$ is a ring iff for every $A, B \in \mathfrak{A}$ the sets $A \cap B$, $A \smile B$ also are members of A. Consequently, A is a ring iff $\mathfrak{A}$ a subring of the ring $(\mathcal{P}(\Omega), \smile, \cap)$.

In what follows we shall deal mainly with the rings and σ-rings. The reader interested only in algebras and σ-algebras can simply replace the word "ring" by the word "algebra".

2.3. Continuity Types of Set Functions

Fix a non-empty family $\mathfrak{A}$ of subsets of Ω, a set $A \in \mathfrak{A}$ and let X be a topological space.

A set function $\mu : \mathfrak{A} \to X$ is said to be
– *(sequentially order) continuous from below at A* if for any increasing sequence (A_n) from $\mathfrak{A}$ with $\lim_n A_n = A$ we have $\lim_n \mu(A_n) = \mu(A)$.
– *(sequentially order) continuous from above at A* if for any decreasing sequence (A_n) from $\mathfrak{A}$ with $\lim_n A_n = A$ we have $\lim_n \mu(A_n) = \mu(A)$.
– *(sequentially) continuous at A* if for any (i.e., not necessarily decreasing or increasing!) sequence (A_n) from $\mathfrak{A}$ with $\lim_n A_n = A$ we have $\lim_n \mu(A_n) = \mu(A)$.
– *(sequentially) right-continuous at A* if for any (i.e., not necessarily decreasing!) sequence (A_n) from $\mathfrak{A}$ such that $\lim_n A_n = A$ and $A \subset A_n$, $n = 1, 2, \ldots$, we have $\lim_n \mu(A_n) = \mu(A)$.
– *(sequentially) left-continuous at A* if for any (i.e., not necessarily increasing!) sequence (A_n) from $\mathfrak{A}$ such that $\lim_n A_n = A$ and $A_n \subset A$, $n = 1, 2, \ldots$, we have $\lim_n \mu(A_n) = \mu(A)$.

In what follows, since no other types of continuities will be considered, the words 'sequentially order' and 'sequentially' in all the above defined notions, will be omitted.

If μ is continuous from above (from below, continuous) *at any set $A \in \mathfrak{A}$*, then we say that μ is *continuous from above (from below, continuous)* on $\mathfrak{A}$.[4]

It is evident that if $\emptyset \in \mathfrak{A}$, then μ is continuous at $\emptyset$ if and only if μ is right-continuous at $\emptyset$.

If $\emptyset \in \mathfrak{A}$ and μ *is continuous from above* at $\emptyset$, then μ will be called *σ-smooth.*

If $\emptyset \in \mathfrak{A}$, then μ is called *exhaustive* (or s-bounded) if for any disjoint sequence $(A_n)_{n\in\mathbb{N}}$ of elements of $\mathfrak{A}$ we have $\lim_n \mu(A_n) = \mu(\emptyset)$.

Lemma 2.4. *If $\emptyset \in \mathfrak{A}$ and μ is continuous at $\emptyset$, then μ is (σ-smooth and) exhaustive.*

Proof. This follows from Lemma 2.2(a). ∎

In what follows, $\mathbb{R}$ will be the set of real numbers with its usual internal operations, order and topology and $\mathbb{R}_+$ will be the set of non-negative real numbers with its usual internal operations, order and topology. We denote also $\overline{\mathbb{R}}_+$ the set $[0, \infty]$ with its usual internal operations, order and topology.

[4]In [4, Sec. 5, p. 279] a set function given on a ring is called *order continuous* if it is continuous from above at $\emptyset$ is our sense.

Let $\mathfrak{A}$ be a ring. Following [4, Sec. 5, p. 279] a set function $\eta : \mathfrak{A} \to \overline{\mathbb{R}}_+$ will be called *a submeasure* if $\eta(\emptyset) = 0$, $\eta(A \cup B) \leq \eta(A) + \eta(B)$ whenever $A, B \in \mathfrak{A}$ and $A \cap B = \emptyset$ and η is increasing, i.e., $\eta(A) \leq \eta(B)$ if $A, B \in \mathfrak{A}$ and $A \subset B$.

Note that if X is a topological space and a function $f : \mathbb{R}_+ \to X$ is "σ-smooth", i.e., has property: for every *decreasing sequence* $(t_n)_{n\in\mathbb{N}}$ of non-negative real numbers with $\lim_n t_n = 0$ the sequence $(f(t_n))_{n\in\mathbb{N}}$ tends to $f(0)$, then f is (right) continuous at 0. The similar statement may not be true for the partially ordered set $(\mathcal{P}(\Omega), \subset)$: a σ-smooth set function $\mu : \mathcal{P}(\Omega) \to X$ may not be (right) continuous at $\emptyset$. The situation is better for the increasing non-negative set functions.

Lemma 2.5. *Let $\mathfrak{A}$ be a σ-ring*[5] *and $\eta : \mathfrak{A} \to \overline{\mathbb{R}}_+$ be a σ-smooth set function which is increasing. Then η is continuous at $\emptyset$.*

Proof. Take a sequence (A_n) of sets from $\mathfrak{A}$ such that $\lim_n A_n = \emptyset$. Put $E_n = \cup_{k=n}^{\infty} A_k$, $n = 1, 2, \ldots$. Since $\mathfrak{A}$ is a σ-ring, $E_n \in \mathfrak{A}$, $n = 1, 2, \ldots$. Moreover, (E_n) is a decreasing sequence of sets such that $A_n \subset E_n$ $n = 1, 2, \ldots$. Since $\lim_n A_n = \emptyset$, we have that $\lim_n E_n = \emptyset$. Since η is σ-smooth, we have also $\eta(E_n) \to \eta(\emptyset)$ when $n \to \infty$. Since η is increasing, we can write

$$\eta(\emptyset) \leq \eta(A_n) \leq \eta(E_n), \; n = 1, 2, \ldots.$$

Then

$$\eta(\emptyset) \leq \liminf_n \eta(A_n) \leq \limsup_n \eta(A_n) \leq \lim_n \eta(E_n) = \eta(\emptyset).$$

Consequently, $\lim_n \eta(A_n) = \eta(\emptyset)$. ∎

Remark 1.

(a)[6] Lemma 2.5 fails when $\mathfrak{A}$ is an algebra of sets. If fact, let Ω be an infinite set. Fix a point $\omega_0 \in \Omega$, denote $\Omega_1 = \Omega \setminus \{\omega_0\}$. Moreover, let $\mathfrak{A}$ be the collection of subsets $A \subset \Omega$ with property: either A or $\Omega \setminus A$ is a finite subset of Ω_1. Then

(1) $\mathfrak{A}$ is an algebra which is not a σ-algebra.

(2) If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of elements of $\mathfrak{A}$ with $\cap_{n=1}^{\infty} A_n = \emptyset$, then there is $n_0 \in \mathbb{N}$ such that $A_n = \emptyset$ for every $n \geq n_0$.

(3) For any topological space X any $\mu : \mathfrak{A} \to X$ is σ-smooth.

(4) Define $\eta : \mathfrak{A} \to \mathbb{R}_+$ as follows: $\eta(\emptyset) = 0$ and $\eta(A) = 1$ whenever $A \neq \emptyset$. Then η is a σ-smooth continuous from below submeasure, which is not *exhaustive* and hence, is not continuous at $\emptyset$.

In fact, (1) and (2) are easy to see; (3) follows from (2). To show (4), take an infinite disjoint sequence (A_n) of non-empty sets from $\mathfrak{A}$. Then $A_n \to \emptyset$, but $\eta(A_n) = 1$, $n = 1, 2, \ldots$. Hence, η is not continuous at $\emptyset$. Note that η is not exhaustive too.

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[5] It is sufficient to require that $\mathfrak{A}$ is a collection of sets such that $\emptyset \in \mathfrak{A}$ and $A_n \in \mathfrak{A}, n = 1, 2, \cdots \Longrightarrow \cup_{n=1}^{\infty} A_n \in \mathfrak{A}$.

[6] *Added in proof* (April, 2003), item (a) is pointed out to us by Z. Lipecki (Poland).

(b) Lemma 2.5 remains true when $\mathfrak{A}$ is a ring of sets and $\eta : \mathfrak{A} \to \overline{\mathbb{R}}_+$ is a σ-smooth *exhaustive submeasure*. This follows from Lemma 2.5 and [4, Theorem 7.2, p. 283] according to which any such η admits an extension to a σ-smooth submeasure $\tilde{\eta} : \mathcal{A} \to \overline{\mathbb{R}}_+$, where $\mathcal{A}$ is the σ-ring generated by $\mathfrak{A}$.

3. From Exhaustivity to σ-Smoothness and to Continuity

As usual, if $(X, \mathcal{T})$ is a topological space, then for a "double sequence" $(x_{m,n})_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ of elements of X and a fixed $x_0 \in X$ we write $\lim_{n,m\to\infty} x_{m,n} = x_0$ if for any neighborhood O of x_0 there is a natural n_o such that $x_{m,n} \in O$ whenever $m > n_o$ and $n > n_o$.

The next proposition contains a criterion of exhaustivity, which is taken from [4, II, 4.1, p. 277], where the case of $\overline{\mathbb{R}}_+$-valued submeasures is considered.

Proposition 3.1. (cf. [4, II, 4.1, p. 277]) *Let $(X, \mathcal{T})$ be a topological space, $\mathfrak{A}$ be a nonvoid collection of sets such that*

$$A, B \in \mathfrak{A},\ A \supset B \Longrightarrow A \setminus B \in \mathfrak{A}$$

and $\mu : \mathfrak{A} \to X$ be a set function. Consider the statements:
(i) *μ is exhaustive;*
(ii) $\lim_{n,m\to\infty} \mu(E_n \setminus E_m) = \mu(\emptyset)$ *for any increasing sequence (E_n) of sets from $\mathfrak{A}$;*
(iii) $\lim_{n,m\to\infty} \mu(E_n \setminus E_m) = \mu(\emptyset)$ *for any decreasing sequence (E_n) of sets from $\mathfrak{A}$.*

Then
(a) (i) $\Longrightarrow$ (ii) *and* (i) $\Longrightarrow$ (iii);
(b) *If $\mathfrak{A}$ is a ring, then* (ii) $\Longrightarrow$ (i);
(c) *If $\mathfrak{A}$ is a σ-ring, then* (iii) $\Longrightarrow$ (i).

Proof.
(a) Let us show e.g., that (i) $\Longrightarrow$ (iii). Suppose that this is not true. Then we can find a decreasing sequence (E_n) of sets from $\mathfrak{A}$ such that $\mu(E_n \setminus E_m) \not\to \mu(\emptyset)$ when $n, m \to \infty$. This implies that for some neighborhood O of $\mu(\emptyset)$ and some sequences (p_n) and (q_n) of natural numbers, such that $n < p_n < q_n < p_{n+1},\ n = 1, 2, \ldots$, we shall have

$$\mu(E_{p_n} \setminus E_{q_n}) \notin O,\ n = 1, 2, \ldots .$$

Therefore we have found a disjoint sequence $(A_n := E_{p_n} \setminus E_{q_n})$ such that $\mu(A_n) \notin O$, $n = 1, 2, \ldots$ But this contradicts the exhaustivity of μ.

The proof of (i) $\Longrightarrow$ (ii) is similar.
(b) Take a disjoint sequence (A_n) of sets from $\mathfrak{A}$. Put $E_n = \cup_{k=1}^n A_k,\ n = 1, 2, \ldots$. Since $\mathfrak{A}$ is a ring, (E_n) is an increasing sequence of sets from $\mathfrak{A}$, such that $E_{n+1} \setminus E_n = A_n,\ n = 1, 2, \ldots$. By assumption now we have $\mu(E_n \setminus E_m) \to \mu(\emptyset)$ when $n, m \to \infty$, so we get $\mu(A_n) \to \mu(\emptyset)$ when $n \to \infty$, i.e., μ is exhaustive.
(c) Take a disjoint sequence (A_n) of sets from $\mathfrak{A}$. Put $E_n = \cup_{k=n}^\infty A_k,\ n = 1, 2, \ldots$. Since $\mathfrak{A}$ is a σ-ring, (E_n) is a decreasing sequence of sets from $\mathfrak{A}$, such

that $E_n \setminus E_{n+1} = A_n,\ n = 1, 2, \ldots$. By assumption we have $\mu(E_n \setminus E_m) \to \mu(\emptyset)$ when $n, m \to \infty$, and we get $\mu(A_n) \to \mu(\emptyset)$ when $n \to \infty$, i.e., μ is exhaustive.

■

Remark 2. Prop. 3.1 (c) may not be true for a ring $\mathfrak{A}$ which is not a σ-ring. E.g., let $\mathfrak{A}$ be the collection of all finite subsets of an infinite countable set Ω. Then $\mathfrak{A}$ is a ring, the non-negative additive set function $A \to \mu(A) := \text{card}\,(A)$ has property (iii), but is not exhaustive.

Let X be a set and $\mathcal{T}$ be a topology in X.

We say that a set $C \subset X$ is $\mathcal{T}$-*sequentially closed* if C contains any limit of every $\mathcal{T}$-convergent sequence consisting of elements of C.

Clearly, if a set $C \subset X$ is $\mathcal{T}$-closed, then C is $\mathcal{T}$-sequentially closed. The converse is true if $\mathcal{T}$ is metrizable, but not in general.

We say that a set $C \subset X$ is $\mathcal{T}$-*neighborhood* of a point $x_0 \in X$ if there is a $\mathcal{T}$-open $U \subset X$ such that $x_0 \in U \subset C$.

We say that a collection $\mathfrak{B}$ of $\mathcal{T}$-*neighborhoods* of a point $x_0 \in X$ is *a fundamental system* of $\mathcal{T}$- neighborhoods for x_0 if for any $\mathcal{T}$-neighborhood O of x_0 there is a set $O_1 \in \mathfrak{B}$ such that $O_1 \subset O$.

The next statement is the main result of the paper.

Theorem 3.2. *Let $(X, \mathcal{T})$ be a topological space, $\mathfrak{A}$ be a collection of sets such that $\emptyset \in \mathfrak{A}$ and $\mu : \mathfrak{A} \to X$ be a set function.*

($\mathcal{H}$) *Let also $\mathcal{T}_o \subset \mathcal{T}$ be a topology in X such that $\mu(\emptyset)$ admits a fundamental system of $\mathcal{T}$-neighborhoods which consists of $\mathcal{T}_o$-sequentially closed sets (this condition is satisfied automatically if $\mathcal{T}_o = \mathcal{T}$ and $(X, \mathcal{T})$ is a regular topological space).*

Assume further that

(1) *$\mu : \mathfrak{A} \to (X, \mathcal{T})$ is exhaustive*
and

(2) *$\mu : \mathfrak{A} \to (X, \mathcal{T}_o)$ is continuous from below.*
Then the following statements are valid:

(a) *If $\mathfrak{A}$ a is a ring*[7]*, then $\mu : \mathfrak{A} \to (X, \mathcal{T})$ is σ-smooth.*

(b) *If $\mathfrak{A}$ a is a σ-ring, then $\mu : \mathfrak{A} \to (X, \mathcal{T})$ is continuous at $\emptyset$.*

Proof.

(a) Take a decreasing sequence (E_n) of sets from $\mathfrak{A}$ which tends to $\emptyset$. Fix a $\mathcal{T}_o$-sequentially closed $\mathcal{T}$-neighborhood O of $\mu(\emptyset)$. Since μ satisfies (1), by Proposition 3.1. (i)$\Longrightarrow$ (iii), we can find a natural number n_o such that $\mu(E_n \setminus E_m) \in O$ whenever $n, m > n_o$. Fix $n > n_o$. Then $(E_n \setminus E_m)_{m \in \mathbb{N}}$ is an increasing sequence of sets from $\mathfrak{A}$ tending to E_n. Since μ satisfies (2), we can write

$$\mu(E_n) = (\mathcal{T}_o) \lim_m \mu(E_n \setminus E_m).$$

[7] *or, merely, $\mathfrak{A}$ has the following property: $A, B \in \mathfrak{A},\ A \supset B \Longrightarrow A \setminus B \in \mathfrak{A}$*

From this, since $\mu(E_n \setminus E_m) \in O$ whenever $m > n_o$ and O is $\mathcal{T}_o$-sequentially closed, we get $\mu(E_n) \in O$ whenever $m > n_o$. Since O is arbitrary (and, by hypothesis, the sets as O constitute a fundamental system of $\mathcal{T}$-neighborhood of $\mu(\emptyset)$), this implies that, $(\mathcal{T})\lim_n \mu(E_n) = \mu(\emptyset)$ and $\mu : \mathfrak{A} \to (X, \mathcal{T})$ is σ-smooth.
(b) Let us assume that μ is not $\mathcal{T}$-continuous at $\emptyset$ and derive a contradiction with the assumption that μ is $\mathcal{T}$-exhaustive. Our assumption (together with $(\mathcal{H})$) implies that there is a $\mathcal{T}_o$-sequentially closed $\mathcal{T}$-neighborhood O of $\mu(\emptyset)$, a (not necessarily decreasing) sequence (E_n) of sets from $\mathfrak{A}$ which tends to $\emptyset$ and *a subsequence* (A_n) of (E_n) such that $\mu(A_n) \notin O$ for all $n \in \mathbb{N}$. By Lemma 2.1 (c) we have that (A_n) also tends to $\emptyset$, i.e., $\cap_{n=1}^{\infty} \cup_{n=k}^{\infty} A_k = \emptyset$. Denote $B_j = \cup_{k=j}^{\infty} A_k$, $j = 1, 2, \ldots$. *Since* $\mathfrak{A}$ *is a* σ*-ring, we have* $B_j \in \mathfrak{A}$, $j = 1, 2, \ldots$. Fix now a natural number n and observe that $(A_n \setminus B_j)_{j\in\mathbb{N}}$ is an increasing sequence of sets from $\mathfrak{A}$ tending to A_n. Since μ is $\mathcal{T}_o$-continuous from below, we can write

$$\mu(A_n) = (\mathcal{T}_o) \lim_j \mu(A_n \setminus B_j).$$

Since O is $\mathcal{T}_o$-sequentially closed and $\mu(A_n) \in X \setminus O$, there is a natural number p_n such that $\mu(A_n \setminus B_{p_n}) \in X \setminus O$. Evidently $p_n > n$. Put now $k_1 = p_1$ and define for any natural number $n > 1$ the natural number k_n by putting $k_n = p_{k_{n-1}}$. Then we shall have $k_n < k_{n+1}$ for all n. We can write

$$\mu(A_{k_n} \setminus B_{k_{n+1}}) \notin O \ \forall\, n \in \mathbb{N}.$$

Consequently, the sequence $(\mu(A_{k_n} \setminus B_{k_{n+1}}))_{n\in\mathbb{N}}$ does not tend to $\mu(\emptyset)$ in the topology $\mathcal{T}$ and this contradicts the fact that μ is $\mathcal{T}$-exhaustive since (see Lemma 2.2 (b)) $(A_{k_n} \setminus B_{k_{n+1}})_{n\in\mathbb{N}}$ is *a disjoint sequence of sets.* ∎

Let us give a typical application of Theorem 3.2.

Corollary 3.3. (cf. [5]) *Let* $(X, \mathcal{T})$ *be a (real or complex) locally convex topological vector space,* $\mathcal{T}_w \subset \mathcal{T}$ *be the weak topology of* $(X, \mathcal{T})$, $\mathfrak{A}$ *be a collection of sets such that* $\emptyset \in \mathfrak{A}$ *and* $\mu : \mathfrak{A} \to X$ *be a set function.*

Assume further that
(1) $\mu : \mathfrak{A} \to (X, \mathcal{T})$ *is exhaustive*
and that
(2) $\mu : \mathfrak{A} \to (X, \mathcal{T}_w)$ *is continuous from below.*
Then the following statements are valid:
(a) *If* $\mathfrak{A}$ *a is a ring, then* $\mu : \mathfrak{A} \to (X, \mathcal{T})$ *is* σ*-smooth.*
(b) *If* $\mathfrak{A}$ *a is a* σ*-ring, then* $\mu : \mathfrak{A} \to (X, \mathcal{T})$ *is continuous at* $\emptyset$.

Proof. Since $(X, \mathcal{T})$ is locally convex, any $x_0 \in X$ admits a fundamental system of convex $\mathcal{T}$-closed neighborhoods. It is well-known that in a locally convex $(X, \mathcal{T})$ any convex $\mathcal{T}$-closed subset of X is weakly closed too. Therefore, $(\mathcal{H})$ from Theorem 3.2 is satisfied in our situation and then the conclusions of the corollary follow from Theorem 3.2 (a, b). ∎

Remark 3. (1) If $\mathfrak{A}$ is a ring and $\eta : \mathfrak{A} \to \overline{\mathbb{R}}_+$ is a σ*-subadditive exhaustive* submeasure, then η is σ-smooth [4, II, Theorem 5.3, p. 280]. This result is not covered by Theorem 3.2 (a).

(2) The method of proof of Theorem 3.2 (b) is taken from the proof of Remark B.1.1 in [7, p. 319–320], where it is used for proving the fact that if μ is an exhaustive σ-smooth additive set function defined on a σ-algebra with values in a complete metric linear space, then μ is continuous at $\emptyset$. We noticed that in the considered proof the additivity and σ-smoothness were used only through continuity from below. This allowed us to prove Theorem 3.2 (b) which does not require a presence of any algebraic structure in the range space.
(3) Theorem 3.2 (b) remains true when $\mathfrak{A}$ is a ring, X is a uniform (uniformizable) topological monoid and $\mu : \mathfrak{A} \to X$ is a *countably additive* exhaustive set function. This can be derived from the extension theorem mentioned in Remark 1 (b) and Theorem 3.2 (b) (see [1] for details).

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