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Continuous Refinable Functions and Self Similarity

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Abstract. Given a finite family of contractive mappings on a metric space and the same number of square matrices, we establish necessary and sufficient conditions such that there is a nontrivial continuous vector field which is refinable relative to these mappings and matrices.

1. Introduction

In this paper we continue our study of refinable functions on invariant sets developed in [3, 7, 9–12] and used in [2, 4, 5, 13] to solve integral equations numerically. Here, we address the question of the existence of a nontrivial continuous solution of the refinement equation. To explain this we recall the setup used in [10].

For $n \in \mathbb{N}$ we denote by $\mathbb{Z}_n$ the set $\{0, 1, \ldots, n-1\}$ and use, in addition, $\mathbb{Z}_\infty = \{0, 1, \ldots\}$. Let $\Phi = \{\phi_j \,:\, j \in \mathbb{Z}_n\}$ be a collection of n contractions on a metric space $X = (X, d)$ with their unique fixed points $\{x_j \,:\, j \in \mathbb{Z}\}$, defined by the equation

$$\phi_j(x_j) = x_j, \quad j \in \mathbb{Z}_n.$$

For $j \in \mathbb{Z}_n$ let $\lambda_j \in (0,1)$ denote the *contractivity constant* of ϕ_j given by

$$\lambda_j := \sup \left\{ \frac{d(\phi_j(x), \phi_j(y))}{d(x,y)} \;:\; x, y \in X,\, x \neq y \right\}.$$

According to [6], there exists a unique closed and bounded set $\Omega \subseteq X$ which satisfies

$$\Omega = \bigcup_{\ell \in \mathbb{Z}_n} \phi_\ell(\Omega), \tag{1}$$

that is, Ω is invariant under the set–valued mapping S defined by setting, for each $Y \subseteq X$,

$$S(Y) := \bigcup_{\ell \in \mathbb{Z}_n} \phi_\ell(Y).$$

Moreover, in [6] it was shown for any compact set $\Gamma \subseteq \Omega$ that

$$\Omega = \lim_{r \to \infty} S^r(\Gamma),$$

where the convergence takes place in the Hausdorff metric and that Ω is itself compact.

To our contractions, we associate a family of n matrices of order $N \times N$,

$$\mathcal{A} = \left\{ \boldsymbol{A}_j \in \mathbb{R}^{N \times N} \ : \ j \in \mathbb{Z}_n \right\}.$$

Definition 1. *A nontrivial continuous vector field* $\boldsymbol{f} : \Omega \to \mathbb{R}^N$ *is called* refinable *with respect to* $(\Phi, \mathcal{A})$ *if*

$$\boldsymbol{A}_j^T \boldsymbol{f} = \boldsymbol{f} \circ \phi_j, \qquad j \in \mathbb{Z}_n, \tag{2}$$

Our goal in this paper is to provide conditions on $(\Phi, \mathcal{A})$ so that equation (2) has a nontrivial *continuous* solution. We use the transpose of matrices in equation (2) to be consistent with the terminology used in previous papers, where the motivation to study (2) is described.

We will consider iterates of the contractions and their action on Ω. For that purpose, we define for $r \in \mathbb{N}$ index vectors $e = (e(j) : j \in \mathbb{Z}_r) \in \mathbb{Z}_n^r$ of *length* $r =: r(e)$ which denotes the number of coefficients of e. The set of all such index vectors will be written as

$$\mathbb{Z}_n^\infty := \bigcup_{r \in \mathbb{N}} \mathbb{Z}_n^r.$$

For any $e \in \mathbb{Z}_n^\infty$ of length r we define the associated composition of the contractions as

$$\phi_e = \phi_{e(0)} \circ \cdots \circ \phi_{e(r-1)}.$$

From (1) it follows readily for any $r \in \mathbb{N}$ that

$$\Omega = \bigcup_{e \in \mathbb{Z}_n^r} \phi_e(\Omega). \tag{3}$$

The family of sets

$$\{\phi_e(\Omega) : e \in \mathbb{Z}_n^r\}$$

gives us a way to say that two points are "close" provided that they are in the same or "neighboring" cells. Although we will not give a formal definition of this "topology", it forms a heuristic foundation for the subsequent analysis.

2. Continuous Refinable Functions: Necessary Conditions

If $\boldsymbol{f} : \Omega \to \mathbb{R}^N$ is a continuous nontrivial vector field satisfying the refinement equation

$$\boldsymbol{A}_j^T \boldsymbol{f} = \boldsymbol{f} \circ \phi_j, \quad j \in \mathbb{Z}_n, \tag{4}$$

it follows for $r \in \mathbb{N}$ that

$$\boldsymbol{A}_e^T \boldsymbol{f} = \boldsymbol{f} \circ \phi_e, \quad e \in \mathbb{Z}_n^r, \tag{5}$$

where

$$\boldsymbol{A}_e := \boldsymbol{A}_{e(r-1)} \cdots \boldsymbol{A}_{e(0)}, \quad e \in \mathbb{Z}_n^r.$$

Observe that the matrices and the contractions are indexed in opposite order.

Lemma 1. *If $\boldsymbol{f}$ is a continuous refinable function with respect to $(\Phi, \mathcal{A})$ and $x \in \Omega$ then $\boldsymbol{f}(x) \neq 0$.*

Proof. Suppose, by contradiction, that there exists an $x \in \Omega$ such that $\boldsymbol{f}(x) = 0$. Since

$$\Omega = \lim_{r \to \infty} S^r\left(\{x\}\right),$$

we can find, for any $y \in \Omega$, a sequence $e_r \in \mathbb{Z}_n^r$, $r \in \mathbb{Z}_\infty$, such that

$$y = \lim_{r \to \infty} \phi_{e_r}(x).$$

The refinement equation for $\boldsymbol{f}$ and its continuity yield the contradiction that

$$\boldsymbol{f}(y) = \lim_{r \to \infty} \left(\boldsymbol{f} \circ \phi_{e_r}\right)(x) = \lim_{r \to \infty} \boldsymbol{A}_{e_r}^T \boldsymbol{f}(x) = 0.$$

Now, our goal is to define the quantities which assist in the study of the refinement equation (2). To this end, we denote for $e \in \mathbb{Z}_n^\infty$ the number of appearances of $j \in \mathbb{Z}_n$ in the components of e by setting

$$\theta_j(e) = \#\{k : k \in \mathbb{Z}_{r(e)},\, e(k) = j\},$$

and assemble them into a vector $\theta(e)$ in $\mathbb{R}^n$, defined by

$$\theta(e) := (\theta_j(e) \ : \ j \in \mathbb{Z}_n).$$

With this quantity in hand, we define, for any $e \in \mathbb{Z}_n^r$, $r \in \mathbb{N}$,

$$\lambda(e) := -\log \lambda^{\theta(e)} \tag{6}$$

and note, for $x, y \in \Omega$ and $e \in \mathbb{Z}_n^\infty$, that

$$d\left(\phi_e(x), \phi_e(y)\right) \leq \lambda^{\theta(e)}\, d(x, y), \tag{7}$$

which yields, in particular, that

$$d\left(\phi_e(\Omega)\right) \leq \lambda^{\theta(e)}\, d\left(\Omega\right), \tag{8}$$

where we use $d(Y) = \sup\{d(x, y) : x, y \in Y\}$ for the *diameter* of a subset Y of X. We also find the quantity newpage

$$\lambda(\Phi) := -\max\{\log \lambda_j \ : \ j \in \mathbb{Z}_n\}$$

useful. Indeed, it satisfies the inequality

$$\lambda(e) \geq r(e)\,\lambda(\Phi), \tag{9}$$

from which it follows that $\lambda(e) \to \infty$ as $r(e) \to \infty$.

Let $\|\cdot\|$ be any norm on $\mathbb{R}^N$. For every matrix $\boldsymbol{A} \in \mathbb{R}^{N\times N}$ and any subspace $V \subseteq \mathbb{R}^N$ we set

$$\|\boldsymbol{A}\|_V := \max\{\|\boldsymbol{A}\boldsymbol{x}\| \ : \ \boldsymbol{x} \in V,\ \|\boldsymbol{x}\| = 1\}.$$

Definition 2. *For any finite collection* $\mathcal{A} = \{\boldsymbol{A}_j : j \in \mathbb{Z}_n\}$ *of* $N \times N$ *matrices, any collection* $\Phi = \{\phi_j : j \in \mathbb{Z}_n\}$ *of contractions and any subspace* W *of* $\mathbb{R}^N$*, we denote the associated* joint spectral radius *by*

$$\rho := \rho(\mathcal{A}, \Phi, W) = \lim_{r\to\infty} \sup\left\{ \left\|\boldsymbol{A}_e^T\right\|_W^{1/\lambda(e)} : e \in \mathbb{Z}_n^r \right\}. \tag{10}$$

The existence of the limit in (10) follows from the additivity of $\lambda(e)$ under "concatenation" of vectors and the multiplicative property of matrix norms. To see this, we call the supremum in (10) ρ_r and set $\mu := \inf\{\rho_r : r \in \mathbb{N}\}$. For any $\varepsilon > 0$ there is an $r \in \mathbb{N}$ such that $\rho_r < \mu + \varepsilon$. For any $p \in \mathbb{N}$ and $e \in \mathbb{Z}_n^p$, we can decompose e as $e = \left(e^j : j \in \mathbb{Z}_L\right)$, where $e^j \in \mathbb{Z}_n^r$, $j \in \mathbb{Z}_{L-1}$, and $e^{L-1} \in \mathbb{Z}_n^q$ for some $q < r$ (here we use the convention that for $p < q$ we have $L = 0$ and $\mathbb{Z}_{L-1} = \emptyset$). Hence,

$$\lambda(e) = \sum_{j\in\mathbb{Z}_L} \lambda\left(e^j\right)$$

and there is a positive constant c such that $\lambda(e) \leq c$ and $\left\|\boldsymbol{A}_e^T\right\| \leq c^{\lambda(e)}$ for all $e \in \mathbb{Z}_n^r$, since this is a finite set of cardinality n^r. Moreover, we have that

$$\begin{aligned} \left\|\boldsymbol{A}_e^T\right\| &\leq \prod_{j\in\mathbb{Z}_L} \left\|\boldsymbol{A}_{e^j}\right\| \leq (\mu+\varepsilon)^{\sum_{j\in\mathbb{Z}_{L-1}} \lambda(e^j)}\, c^{\lambda(e^{L-1})} \\ &= (\mu+\epsilon)^{\lambda(e)} \left(\frac{c}{\mu+\epsilon}\right)^c. \end{aligned}$$

Since $\lambda(e) \to \infty$ as $r(e) \to \infty$, we obtain that

$$\limsup_{r\to\infty} \rho_r \leq \mu + \varepsilon$$

which proves the result.

Theorem 1. *If* $\boldsymbol{f}$ *is a refinable continuous vector field then*

$$\rho(\mathcal{A}, \Phi, W) < 1, \tag{11}$$

where

$$W := \operatorname{span}\{\boldsymbol{f}(x) - \boldsymbol{f}(y) : x, y \in \Omega\}. \tag{12}$$

Proof. Since W is finite dimensional, there exist $s \in \mathbb{N}$ and $x_j, y_j \in \Omega$, $j \in \mathbb{Z}_s$, such that any $\boldsymbol{w} \in W$ can be uniquely written as

$$\boldsymbol{w} = \sum_{j\in\mathbb{Z}_s} w_j \left(\boldsymbol{f}\left(x_j\right) - \boldsymbol{f}\left(y_j\right)\right).$$

We choose a constant $c > 0$ such that

$$\sum_{j\in\mathbb{Z}_s} |w_j| \le c\,\|\boldsymbol{w}\|. \tag{13}$$

By the uniform continuity of $\boldsymbol{f}$ there exists $\delta \in (0,1)$, such that whenever $d(x,y) \le \delta$ we have that $\|\boldsymbol{f}(x) - \boldsymbol{f}(y)\| < c/2$. Likewise, we can find an integer $r \in \mathbb{N}$ such that $\lambda(e) \ge -\log\delta + \omega$ whenever $r(e) \ge r$, where $\omega := \log d(\Omega)$. Therefore, by (8), we have for any $e \in \mathbb{Z}_n^\infty$ with $r(e) \ge r$ that

$$d\big(\phi_e(\Omega)\big) \le \lambda^{\theta(e)}\, d(\Omega) = e^{-\lambda(e)+\omega} \le e^{\log\delta} = \delta.$$

Consequently, our choice of δ yields for any $x, y \in \Omega$ that

$$\|(\boldsymbol{f}\circ\phi_e)(x) - (\boldsymbol{f}\circ\phi_e)(y)\| < c/2, \tag{14}$$

and it follows from the refinement equation (5), (13) and (14) that $\|\boldsymbol{A}_e^T\|_W < 1/2$ whenever $r(e) \ge r$. In addition, there exists a constant $m > 0$ such that $\lambda(e) \le -m\log\delta$ for all $e \in \mathbb{Z}_n^r$. Therefore, we obtain for any $e \in \mathbb{Z}_n^r$ that

$$\big\|\boldsymbol{A}_e^T\big\|_W^{1/\lambda(e)} \le 2^{-1/\lambda(e)} \le 2^{1/(m\,\log\delta)} =: \gamma < 1. \tag{15}$$

Now, for any $p \in \mathbb{N}$, we decompose $e \in \mathbb{Z}_n^p$ as before into a concatenation of vectors

$$e = \left(e^j : j \in \mathbb{Z}_L\right), \quad e^j \in \mathbb{Z}_n^r,\ j \in \mathbb{Z}_{L-1}, \quad e^{L-1} \in \mathbb{Z}_n^q,\ q < r,$$

yielding that $\lambda\left(e^j\right) \ge -\log\delta + \omega$, $j \in \mathbb{Z}_{L-1}$. Thus, we conclude that

$$\big\|\boldsymbol{A}_e^T\big\|_W \le \big\|\boldsymbol{A}_{e^{L-1}}^T\big\|_W \prod_{j\in\mathbb{Z}_{L-1}} \gamma^{\lambda(e^j)} = \big\|\boldsymbol{A}_{e^{L-1}}^T\big\|_W \prod_{j\in\mathbb{Z}_{L-1}} \gamma^{\lambda(e)-\lambda(e^{L-1})},$$

and therefore setting

$$\mu := \gamma^{-m\log\delta}\ \max\left\{\big\|\boldsymbol{A}_e^T\big\|_W \ :\ e \in \mathbb{Z}_n^\infty,\ r(e) < r\right\},$$

we obtain that

$$\begin{aligned}\big\|\boldsymbol{A}_e^T\big\|_W^{1/\lambda(e)} &\le \big\|\boldsymbol{A}_{e^{L-1}}^T\big\|_W^{1/\lambda(e)} \gamma^{1-\lambda\left(e^{L-1}\right)/\lambda(e)} \\ &\le \big\|\boldsymbol{A}_{e^{L-1}}^T\big\|_W^{1/\lambda(e)} \gamma^{1+(\log\delta-\omega)/\lambda(e)} \le \gamma\,\mu^{1/\lambda(e)}.\end{aligned}$$

Since $\lim_{r(e)\to\infty}\lambda(e) = \infty$ and $\gamma < 1$, the claim (11) follows from this estimate. ■

We finish this section by recalling a few further consequences of the existence of a refinable function which a patterned according to [1, 7, 8].

Proposition 1. *If* $\boldsymbol{f}$ *is a continuous* $(\mathcal{A}, \Phi)$*–refinable function then*

1. *Each matrix* $\boldsymbol{A}_j^T$, $j \in \mathbb{Z}_n$, *has the eigenvalue one, that is, we have for any* $j \in \mathbb{Z}_n$ *that*
$$\boldsymbol{A}_j^T \boldsymbol{u}_j = \boldsymbol{u}_j, \quad \boldsymbol{u}_j = \boldsymbol{f}(x_j). \tag{16}$$
2. *For any* $e, e' \in \mathbb{Z}_n^\infty$ *and* $j, j' \in \mathbb{Z}_n$ *such that* $\phi_e(x_j) = \phi_{e'}(x_{j'})$, *the* compatibility condition
$$\boldsymbol{A}_e^T \boldsymbol{u}_j = \boldsymbol{A}_{e'}^T \boldsymbol{u}_{j'}. \tag{17}$$
must be satisfied.
3. *For any* $j, k \in \mathbb{Z}_n$ *we have that*
$$\boldsymbol{A}_j^T \boldsymbol{u}_k - \boldsymbol{A}_k^T \boldsymbol{u}_j \in W. \tag{18}$$

Proof. Since $x_j = \phi_j(x_j)$ it follows that
$$\boldsymbol{f}(x_j) = \boldsymbol{f}(\phi_j(x_j)) = \boldsymbol{A}_j^T \boldsymbol{f}(x_j),$$
which verifies (16). If, on the other hand, $\phi_e(x_j) = \phi_{e'}(x_{j'})$, then
$$\boldsymbol{A}_e^T \boldsymbol{f}(x_j) = \boldsymbol{f}\big(\phi_e(x_j)\big) = \boldsymbol{f}\big(\phi_{e'}(x_{j'})\big) = \boldsymbol{A}_{e'}^T \boldsymbol{f}(x_{j'}),$$
which also completes the proof of (17). The last assumption (18) follows from the fact that for $j, k \in \mathbb{Z}_n$ we see that
$$\boldsymbol{A}_j^T \boldsymbol{u}_k - \boldsymbol{A}_k^T \boldsymbol{u}_j = \boldsymbol{A}_j^T \boldsymbol{f}(x_k) - \boldsymbol{A}_k^T \boldsymbol{f}(x_j) = \boldsymbol{f}\big(\phi_j(x_k)\big) - \boldsymbol{f}\big(\phi_k(x_j)\big),$$
which obviously belongs to W. ■

Remark 1. The argument which led to the compatibility conditions (17) can be improved. Specifically, we note that whenever there are $e, e' \in \mathbb{Z}_n^r$, $r \in \mathbb{N}$, and $x, x' \in \Omega$ such that $\phi_e(x) = \phi_{e'}(x')$, then
$$\boldsymbol{A}_e^T \boldsymbol{f}(x) = \boldsymbol{A}_{e'}^T \boldsymbol{f}(x').$$
In particular, any point in the intersection of two cells $\phi_j(\Omega) \cap \phi_k(\Omega)$, $j, k \in \mathbb{Z}_n$, $j \neq k$, defines another compatibility condition!

We define the vector space
$$V := \operatorname{span}\big\{\boldsymbol{f}(x) : x \in \Omega\big\}$$
which is an $\mathcal{A}$–invariant subspace of $\mathbb{R}^N$, since $\boldsymbol{f}$ is refinable. That is, we have for every $j \in \mathbb{Z}_n$ that $\boldsymbol{A}_j^T V \subseteq V$. In fact, V is the *smallest* $\mathcal{A}$–invariant subspace T of $\mathbb{R}^N$ for which there exists an $y \in T$ such that $\boldsymbol{f}(y) \in T$. Indeed, for any $x \in \Omega$ choose a sequence $\{e^r : r \in \mathbb{N}\} \in \mathbb{Z}_n^\infty$ such that
$$x = \lim_{r \to \infty} \phi_{e^r}(y).$$
We conclude by the continuity and refinability of $\boldsymbol{f}$ that
$$\boldsymbol{f}(x) = \lim_{r \to \infty} \boldsymbol{A}_{e^r}^T \boldsymbol{f}(y).$$

Since T is closed, being a finite dimensional space, and $\mathcal{A}$–invariant, we conclude that $\boldsymbol{f}(x) \in T$ and hence $V \subseteq T$. Specifying this observation to the choice $x = x_j$, $j \in \mathbb{Z}_n$, and using the notation from (16), we thus get for $j \in \mathbb{Z}_n$ that

$$V = \operatorname{span}\{\boldsymbol{A}_e^T \boldsymbol{u}_j : e \in \mathbb{Z}_n^\infty\} = \operatorname{span}\{\boldsymbol{A}_e^T \boldsymbol{u}_k : e \in \mathbb{Z}_n^\infty,\ k \in \mathbb{Z}_n\}. \tag{19}$$

Now, we can relate V and W in the following fashion.

Proposition 2. *If $\boldsymbol{f}$ is continuous refinable function and $j \in \mathbb{Z}_n$ then*

$$V = W \oplus \operatorname{span}\{\boldsymbol{u}_j\} \tag{20}$$

and consequently

$$V = W + \operatorname{span}\{\boldsymbol{u}_k : k \in \mathbb{Z}_n\}. \tag{21}$$

Proof. It is clear that $W \subseteq V$. Now, fix $j \in \mathbb{Z}_n$ and note that the equation $\boldsymbol{A}_j^T \boldsymbol{u}_j = \boldsymbol{u}_j$ yields the inequality $\rho(\mathcal{A}, \Phi, \operatorname{span}\{\boldsymbol{u}_j\}) \geq 1$, from which it follows, in view of Theorem 1, that $\boldsymbol{u}_j \in V \setminus W$. In view of (19), it suffices to prove for any $e \in \mathbb{Z}_n^\infty$ that $\boldsymbol{A}_e^T \boldsymbol{u}_j - \boldsymbol{u}_j \in W$, which we prove by induction on $r := r(e)$. For $r \in \mathbb{N}$ and $k \in \mathbb{Z}_n$ we have that

$$\boldsymbol{A}_k^T \boldsymbol{A}_e^T \boldsymbol{u}_j - \boldsymbol{u}_j = \boldsymbol{A}_k^T (\boldsymbol{A}_e^T \boldsymbol{u}_j - \boldsymbol{u}_j) + \boldsymbol{A}_k^T (\boldsymbol{u}_j - \boldsymbol{u}_k) + \boldsymbol{u}_k - \boldsymbol{u}_j.$$

The first term belongs to W by the induction hypothesis, the second is in W since W is $\mathcal{A}$–invariant and the third term is in W by its definition (12). ■

Remark 2. Note that, in contrast to (20), (21) is *not* a direct sum in general. For example, the two matrices given later in (31) have $\boldsymbol{u}_0 = [1,0]^T$ and $\boldsymbol{u}_1 = [0,1]^T$, so that $\operatorname{span}\{\boldsymbol{u}_0, \boldsymbol{u}_1\} = \mathbb{R}^2 \supseteq V$ and $W \neq \{0\}$.

Finally, we observe that there exists a linear combination of the components of a refinable vector field which is the constant function.

Corollary 1. *If $\boldsymbol{f}$ is a continuous refinable vector field then there exists $\boldsymbol{y} \in \mathbb{R}^N$ such that $\boldsymbol{y}^T \boldsymbol{f} = 1$.*

Proof. It follows from (20) that there exists $\boldsymbol{y} \in \mathbb{R}^N$ such that $\boldsymbol{y}^T \boldsymbol{x} = 0$, $\boldsymbol{x} \in W$ and $\boldsymbol{y}^T \boldsymbol{u}_0 = 1$. Therefore, it follows for any $x, y \in \Omega$ that $0 = \boldsymbol{y}^T (\boldsymbol{f}(x) - \boldsymbol{f}(y))$ which means that $\boldsymbol{y}^T \boldsymbol{f}(x) = \boldsymbol{y}^T \boldsymbol{f}(x_0) = 1$, $x \in \Omega$. ■

3. Continuous Refinable Functions: Sufficient Conditions

We now show that the necessary conditions on the matrices $\mathcal{A}$ and the contractions in Φ are also sufficient for the existence of a continuous $(\mathcal{A}, \Phi)$–refinable function under additional conditions on Φ. To this end, we explore the consequences of the necessary conditions for the existence of a refinable function that we derived above.

Assumption 1. *Let $\mathcal{A} = \{\boldsymbol{A}_j : j \in \mathbb{Z}_n\}$ be a family of $N \times N$ matrices such that there exist nonzero vectors $\{\boldsymbol{u}_j : j \in \mathbb{Z}_n\}$ such that*

$$\boldsymbol{u}_j = \boldsymbol{A}_j^T \boldsymbol{u}_j, \quad j \in \mathbb{Z}_n, \tag{22}$$

with the following properties.

1. *If* $e, e' \in \mathbb{Z}_n^\infty$, $j, j' \in \mathbb{Z}_n$ *and* $\phi_e(x_j) = \phi_{e'}(x_{j'})$, *then*
$$\boldsymbol{A}_e^T \boldsymbol{u}_j = \boldsymbol{A}_{e'}^T \boldsymbol{u}_{j'}. \tag{23}$$
2. *There exists a subspace* $W \subseteq \mathbb{R}^N$ *such that for any* $j, k \in \mathbb{Z}_n$
$$\boldsymbol{u}_j - \boldsymbol{u}_k \in W, \quad \boldsymbol{A}_k^T \boldsymbol{u}_j - \boldsymbol{A}_j^T \boldsymbol{u}_k \in W. \tag{24}$$
and
$$V = U + W, \tag{25}$$
where $U = \operatorname{span}\{\boldsymbol{u}_j : j \in \mathbb{Z}_n\}$ *and* $V = \operatorname{span}\{\boldsymbol{A}_e^T \boldsymbol{u}_j : e \in \mathbb{Z}_n^\infty,\ j \in \mathbb{Z}_n\}$.
3. *The joint spectral radius* $\rho(\mathcal{A}, \Phi, W)$ *is less than one.*

The remainder of this section is dedicated to the construction of a refinable vector field $\boldsymbol{f}$ provided that Assumption 1 holds true and that Ω satisfies certain topological constraints.

Lemma 2. *If Assumption* 1 *is satisfied then the space* W *is* $\mathcal{A}$*–invariant, that is,*
$$\boldsymbol{A}_j^T W \subseteq W, \quad j \in \mathbb{Z}_n.$$

Proof. Choose $\boldsymbol{w} \in W \subseteq V$, $j \in \mathbb{Z}_n$ and write
$$\boldsymbol{A}_j^T \boldsymbol{w} = \boldsymbol{w}' + \sum_{k \in \mathbb{Z}_n} c_k \, \boldsymbol{u}_k$$
for some $\boldsymbol{w}' \in W$ and $c = (c_k : k \in \mathbb{Z}_n) \in \mathbb{R}^n$. Since $\boldsymbol{u}_k - \boldsymbol{u}_0 \in W$ for $k \in \mathbb{Z}_n$, we can find $d_0 \in \mathbb{R}$ and $\widetilde{\boldsymbol{w}} \in W$ such that $\boldsymbol{A}_j^T \boldsymbol{w} = \widetilde{\boldsymbol{w}} + d_0 \, \boldsymbol{u}_0$. Therefore, for every $r \in \mathbb{N}$ we have that
$$|d_0| \, \|\boldsymbol{u}_0\| \leq \left\| (\boldsymbol{A}_0^T)^r \boldsymbol{A}_j^T \boldsymbol{w} \right\| + \left\| (\boldsymbol{A}_0^T)^r \boldsymbol{w} \right\| .$$
Each term on the right hand side goes to zero as $r \to \infty$ since $\rho(\mathcal{A}, \Phi, W) < 1$. Consequently, $d_0 = 0$ and we conclude that $\boldsymbol{A}_j^T \boldsymbol{w} \in W$, $j \in \mathbb{Z}_n$, thereby proving the result. ∎

We define the *grid*
$$G = \left\{ \phi_e(x_j) : e \in \mathbb{Z}_n^\infty,\ j \in \mathbb{Z}_n \right\},$$
which is the dense subset of Ω generated by applying all the iterates of Φ to the fixed points of Φ. On this set, we define the function $\boldsymbol{f}$ by the equation
$$\boldsymbol{f}\,(\phi_e(x_j)) := \boldsymbol{A}_e^T \boldsymbol{u}_j, \quad e \in \mathbb{Z}_n^\infty, \quad j \in \mathbb{Z}_n. \tag{26}$$
Property 1) in Assumption 1 implies that $\boldsymbol{f}$ is well-defined on G. Moreover, for any $e \in \mathbb{Z}_n^\infty$, $j \in \mathbb{Z}_n$, and $x = \phi_e(x_j) \in G$ we have that
$$(\boldsymbol{f} \circ \phi_k)(x) = (\boldsymbol{f} \circ \phi_k \circ \phi_e)(x) = \boldsymbol{A}_k^T \boldsymbol{A}_e^T \, \boldsymbol{u}_j = \boldsymbol{A}_k^T \boldsymbol{f}(x), \quad k \in \mathbb{Z}_n.$$

In other words, $\boldsymbol{f}$ is refinable on G.

Lemma 3. *If Assumption 1 is satisfied then we have, for any $x, y \in G$, that $\boldsymbol{f}(x) - \boldsymbol{f}(y) \in W$.*

Proof. We first prove that

$$\boldsymbol{A}_e^T \boldsymbol{u}_j - \boldsymbol{u}_j \in W, \;\; e \in \mathbb{Z}_n^r, \;\; r \in \mathbb{N}, \tag{27}$$

which is done by induction on r. For $r = 1$ we have that

$$\boldsymbol{A}_k^T \boldsymbol{u}_j - \boldsymbol{u}_j = (\boldsymbol{A}_k^T \boldsymbol{u}_j - \boldsymbol{A}_j^T \boldsymbol{u}_k) + \boldsymbol{A}_j^T (\boldsymbol{u}_k - \boldsymbol{u}_j).$$

The first term on the right hand side is in W because of 2) in Assumption 1, while the second term belongs to W by 2) of Assumption 1 and Lemma 2. To advance the induction hypothesis for $r > 1$, we consider, for any $k \in \mathbb{Z}_n$ and $e \in \mathbb{Z}_n^r$, the identity

$$\boldsymbol{A}_k^T \boldsymbol{A}_e^T \boldsymbol{u}_j - \boldsymbol{u}_j = \boldsymbol{A}_k^T (\boldsymbol{A}_e^T \boldsymbol{u}_j - \boldsymbol{u}_j) + \boldsymbol{A}_k^T \boldsymbol{u}_j - \boldsymbol{u}_j.$$

The induction hypothesis and Lemma 2 imply that the first term is in W, while the second term belongs to W due to the case $r = 1$. This completes the proof of (27).

Now, let $x = \phi_e(x_j)$ and $y = \phi_{e'}(x_k)$ be given, where $j, k \in \mathbb{Z}_n$, $e, e' \in \mathbb{Z}_n^\infty$, and compute

$$\begin{aligned} \boldsymbol{f}(x) - \boldsymbol{f}(y) &= \boldsymbol{A}_e^T \boldsymbol{u}_j - \boldsymbol{A}_{e'}^T \boldsymbol{u}_k \\ &= \boldsymbol{A}_e^T (\boldsymbol{u}_j - \boldsymbol{u}_k) + (\boldsymbol{A}_e^T \boldsymbol{u}_k - \boldsymbol{u}_k) - (\boldsymbol{A}_{e'}^T \boldsymbol{u}_k - \boldsymbol{u}_k). \end{aligned}$$

The first term belongs to W by Lemma 2 and the other two by (27) which verifies our claim. ∎

Lemma 4. *If Assumption 1 is satisfied then there is a $\boldsymbol{y} \in \mathbb{R}^N$ such that $\boldsymbol{y}^T \boldsymbol{f} = 1$.*

Proof. By 2) of Assumption 1, we have that $V = \operatorname{span}\{\boldsymbol{u}_0\} \oplus W$. We choose a $\boldsymbol{y} \perp W$ such that $\boldsymbol{y}^T \boldsymbol{u}_0 = 1$. For any $e \in \mathbb{Z}_n^\infty$ and $j \in \mathbb{Z}_n$, Lemma 3 and the definition of $\boldsymbol{f}$ imply that $\boldsymbol{A}_e^T \boldsymbol{u}_j - \boldsymbol{u}_0 \in W$ and so $\boldsymbol{y}^T \boldsymbol{A}_e^T \boldsymbol{u}_j = 1$, that is, $\boldsymbol{y}^T \boldsymbol{f} = 1$ on G. ∎

Lemma 5. *If Assumption 1 is satisfied then there exists a positive constant c such that for all $x \in G$*

$$\|\boldsymbol{f}(x)\| \le c.$$

Proof. By 3) of Assumption 1 there exists a number $r \in \mathbb{N}$ such that, for any $\boldsymbol{w} \in W$ and $\widetilde{e} \in \mathbb{Z}_n^r$ we have that

$$\left\|\boldsymbol{A}_{\widetilde{e}}^T \boldsymbol{w}\right\| \le \frac{1}{2} \|\boldsymbol{w}\|.$$

Choose a positive constant c such that for all $j' \in \mathbb{Z}_n$ and $e' \in \mathbb{Z}_n^r$ we have that $\|\boldsymbol{u}_{j'}\| \le c$ and $\left\|\boldsymbol{A}_{e'}^T \boldsymbol{u}_{j'} - \boldsymbol{u}_{j'}\right\| \le c$. For any $x \in G$, written in the form $x = \phi_e(x_j)$

for some $j \in \mathbb{Z}_n$ and $e \in \mathbb{Z}_n^\infty$, we "pad" e with j a sufficient number of times so that the composite vector $e^* := (e, j, \ldots, j)$ is in $\mathbb{Z}_n^{mr}$ for some (least) value of $m \in \mathbb{N}$, and write $e^* = (e^k : k \in \mathbb{Z}_m)$, where $e^k \in \mathbb{Z}_n^r$, $k \in \mathbb{Z}_m$. By the definition of $\boldsymbol{f}$ and by our choice of e^* and x we have that $\boldsymbol{f}(x) = \boldsymbol{A}_{e^*}^T \boldsymbol{u}_j$. Therefore, we obtain the estimate

$$\begin{aligned} \|\boldsymbol{f}(x)\| &= \|\boldsymbol{u}_j\| + \sum_{k \in \mathbb{Z}_m} \|\boldsymbol{A}_{e^0}^T \cdots \boldsymbol{A}_{e^{k-1}}^T (\boldsymbol{A}_{e^k} \boldsymbol{u}_j - \boldsymbol{u}_j)\| \\ &\le \|\boldsymbol{u}_j\| + \sum_{k \in \mathbb{Z}_m} 2^{-k} \|\boldsymbol{A}_{e^k} \boldsymbol{u}_j - \boldsymbol{u}_j\| \le 2c, \end{aligned}$$

which completes the proof. ■

For $x \in \Omega$ we denote the closed ball of radius t with center x by

$$B(x,t) := \{y \in \Omega : d(x,y) \le t\}.$$

Definition 3. *We say that the family Φ of contractive mappings has finite density provided that there exists an integer $M \in \mathbb{N}$ such that for any $e \in \mathbb{Z}_n^\infty$, and $x \in \phi_e(\Omega)$*

$$\#\{e' : B(x, \lambda^{\theta(e)}) \cap \phi_{e'}(\Omega) \ne \emptyset,\ r(e') = r(e)\} \le M. \tag{28}$$

The set Ω is called locally connected if for any $x \in \Omega$, any ball $B(x,r) \subseteq X$, $r > 0$, and any $x' \in B(x,r) \cap \Omega$ there exists a continuous curve $\psi : [0,1] \to B(x,r) \cap \Omega$ such that $\psi(0) = x$, $\psi(1) = x'$.

Definition 4. *The family Φ of contractive mappings with associated invariant set Ω is said to have fix point intersections if for any $e, e' \in \mathbb{Z}_n^\infty$, $e \ne e'$, $r(e) = r(e')$, such that $\phi_e(\Omega) \cap \phi_{e'}(\Omega) \ne \emptyset$, then there exist $j, k \in \mathbb{Z}_n$ such that*

$$\phi_e(x_j) = \phi_{e'}(x_k). \tag{29}$$

Lemma 6. *If Assumption 1 is satisfied, Φ has finite density, Ω is locally connected and has fix point intersections then the function $\boldsymbol{f}$ is continuous on G.*

Proof. Fix any σ with $\rho(\mathcal{A}, \Phi, W) < \sigma < 1$. There exists a constant $c_\sigma > 0$ such that for any $e \in \mathbb{Z}_n^\infty$ we have that

$$\left\|A_e^T\right\|_W \le c_\sigma\, \sigma^{\lambda(e)}.$$

Let $x \in \phi_e(\Omega) \cap G$, $e \in \mathbb{Z}_n^r$, $r \in \mathbb{N}$, and $x' \in B(x, \lambda^{\theta(e)}) \cap G$. If $x' \in \phi_e(\Omega)$, then we choose $y, y' \in \Omega \cap G$ such that $x = \phi_e(y)$, $x' = \phi_e(y')$ and obtain that

$$\begin{aligned} \|\boldsymbol{f}(x) - \boldsymbol{f}(x')\| &= \|(\boldsymbol{f} \circ \phi_e)(y) - (\boldsymbol{f} \circ \phi_e)(y')\| \\ &= \|\boldsymbol{A}_e^T (\boldsymbol{f}(y) - \boldsymbol{f}(y'))\| \le 2c\, c_\sigma\, \sigma^r \end{aligned}$$

by Lemma 5.

Now, suppose that $x' \notin \phi_e(\Omega)$ and let $\psi : [0,1] \to B(x, \lambda^{\theta(e)}) \cap \Omega$, $\psi(0) = x$, $\psi(1) = x'$, be a curve that connects x to x'. Set $e^0 = e$ and $t_0 = \max\{t \in [0,1] : \psi(t) \in \phi_e(\Omega)\}$. This maximum exists since $\phi_e(\Omega)$ is closed and ψ is continuous. Since $x' \notin \phi_e(\Omega)$ we have that $t_0 < 1$. Moreover, $y_1 := \psi(t_0)$ must belong to $\phi_{e^0}(\Omega) \cap \phi_{e^1}(\Omega)$ for some $e^1 \in \mathbb{Z}_n^r \setminus \{e^0\}$. Indeed, to prove this claim, we choose any sequence $t^{(j)} \in (t_0, 1]$ with $t^{(j)} \to t_0$ for $j \to \infty$ and conclude, by the definition of t_0, for any $j \in \mathbb{N}$ that $\psi(t^{(j)}) \notin \phi_e(\Omega)$. There is a subsequence $s^{(j)}$ of $t^{(j)}$ and some fixed $e^1 \in \mathbb{Z}_n^r \setminus \{e^0\}$ such that for all $j \in \mathbb{N}$ we have

$$\psi(s^{(j)}) \in \phi_{e^1}(\Omega).$$

By the continuity of ϕ_{e^1} we can then conclude that

$$\psi(t_0) = \lim_{j\to\infty} \psi(s^{(j)}) \in \phi_{e^1}(\Omega).$$

Hence, $y_1 := \phi_{e^0}(z_0) = \phi_{e^1}(\widehat{z}_1)$ for $z_0, \widehat{z}_1 \in \Omega$. If $x' \notin \phi_{e^1}(\Omega)$, we proceed in the same way by setting $t_1 = \max\{t \in [t_0, 1] \ : \ \psi(t) \in \phi_{e^1}(\Omega)\}$, until, by the finiteness condition (28), we conclude that (independently of e and r) there exist points $y_j \in B(x, \lambda^{\theta(e)})$, $j \in \mathbb{Z}_m$, $m \le M$, such that

$$y_j = \phi_{e^j}(z_j) = \phi_{e^{j+1}}(\widehat{z}_j), \quad z_j, \hat{z}_j \in \Omega, \quad j \in \mathbb{Z}_m,$$

and $x' = y_m$. Since each z_j and $\hat{z}_j$ corresponds to an intersection of $\phi_{e^j}(\Omega)$ and $\phi_{e^{j+1}}(\Omega)$ where $r(e^j) = r(e^{j+1})$ and since Ω has fix point intersections, we can choose $z_j, \hat{z}_j \in \{x_k \ : \ k \in \mathbb{Z}_n\}$, $j \in \mathbb{Z}_m$. Hence, noting that $r(e^j) = r$, $j \in \mathbb{Z}_{m+1}$, we obtain the estimate

$$\begin{aligned}
&\|\boldsymbol{f}(x) - \boldsymbol{f}(x')\| \le \sum_{j\in\mathbb{Z}_m} \|\boldsymbol{f}(y_{j+1}) - \boldsymbol{f}(y_j)\| \\
&= \sum_{j\in\mathbb{Z}_m} \|(\boldsymbol{f}\circ\phi_{e^{j+1}})(z_{j+1}) - (\boldsymbol{f}\circ\phi_{e^{j+1}})(\widehat{z}_j)\| \\
&= \sum_{j\in\mathbb{Z}_m} \|\boldsymbol{A}^T_{e^{j+1}}(\boldsymbol{f}(z_{j+1}) - \boldsymbol{f}(\widehat{z}_j))\| \le \sum_{j\in\mathbb{Z}_m} \|\boldsymbol{A}^T_{e^{j+1}}\|_W \|\boldsymbol{f}(z_{j+1}) - \boldsymbol{f}(\widehat{z}_j)\| \\
&\le 2\, c\, c_\sigma \sum_{j\in\mathbb{Z}_m} \sigma^{\lambda(e)} \le 2\, c\, c_\sigma\, (M+1)\, \sigma^{r(e)\,\lambda(\Phi)}
\end{aligned}$$

where we used (9) in the last inequality. From this estimate, the continuity of $\boldsymbol{f}$ on G follows readily. ∎

Because of Lemma 5, Lemma 6 and the density of the grid G in Ω, we can extend $\boldsymbol{f}$ to a continuous function on *all* of Ω and obtain the following result.

Theorem 2. *If Assumption* 1 *is satisfied,* Φ *has finite density,* Ω *is locally connected and has fix point intersections then there exists a nontrivial continuous refinable function* $\boldsymbol{f}$.

We close this sections with some remarks about the hypotheses on Φ in Theorem 2. In general, without the fix point intersection property the existence of a continuous refinable function cannot be decided from properties of the refinement matrices alone. To elaborate on this remark, we observe that $\Omega = [0, 1]$ is the invariant set corresponding to the two *strictly decreasing* contractions $\phi_0(x) = (1 - x)/2$ and $\phi_1(x) = (2 - x)/2$ which map $[0, 1]$ onto $[0, 1/2]$ and $[1/2, 1]$, respectively. The fixed points of the mappings are $x_0 = 1/3$ and $x_1 = 2/3$ and since they lie *strictly inside* their respective cells, the compatibility conditions in (23) are void. Now, observe that the function $\boldsymbol{f}(x) = [1-x, x]^T$ is refinable with respect to Φ and the matrices

$$\boldsymbol{A}_0 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

On the other hand, the choice

$$\boldsymbol{A}_0 = \begin{bmatrix} 1 & 0 \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}, \quad \boldsymbol{A}_1 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

satisfies all the conditions of Theorem 1, but does not admit a continuous refinable function. To show this, we first note that the two contractions

$$\psi_0(x) := (\phi_0 \circ \phi_1)(x) = \frac{x}{4}, \quad \psi_1(x) := (\phi_1 \circ \phi_0)(x) = \frac{3}{4} + \frac{x}{4}, \quad x \in [0,1],$$

have the fixed points $y_0 = 0$ and $y_1 = 1$, respectively. Therefore, we have for any $x \in [0,1]$ that

$$\lim_{k\to\infty} \psi_0^k(x) = 0, \quad \lim_{k\to\infty} \psi_1^k(x) = 1.$$

We define the two matrices

$$\boldsymbol{B}_0 = \boldsymbol{A}_1^T \boldsymbol{A}_0^T = \begin{bmatrix} \frac{2}{3} & \frac{2}{9} \\ \frac{1}{3} & \frac{7}{9} \end{bmatrix}, \quad \boldsymbol{B}_1 = \boldsymbol{A}_0^T \boldsymbol{A}_1^T = \begin{bmatrix} \frac{7}{9} & \frac{1}{3} \\ \frac{2}{9} & \frac{2}{3} \end{bmatrix},$$

conclude, from the assumption that there exists a continuous refinable function $\boldsymbol{f}$, the equations

$$\boldsymbol{f}(0) = \lim_{k\to\infty} \boldsymbol{f}\big(\psi_0^k(x_0)\big) = \lim_{k\to\infty} \boldsymbol{B}_0^k \boldsymbol{f}(x_0) = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{3}{5} \end{bmatrix}$$

and likewise we have that $\boldsymbol{f}(1) = (\frac{3}{5}, \frac{2}{5})^T$. However, it follows from the refinement equation that

$$\begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix} = \boldsymbol{A}_0^T \boldsymbol{f}(0) = \boldsymbol{f}\,[\,\tfrac{1}{2}\,] = \boldsymbol{A}_1^T \boldsymbol{f}(1) = \begin{bmatrix} \frac{2}{5} \\ \frac{3}{5} \end{bmatrix},$$

a contradiction.

4. Existence of Continuous Refinable Functions: A Complete Characterization

In this section we combine the observations made in the two preceding sections and give a complete characterization of the existence of a continuous refinable function corresponding to a family $\mathcal{A}$ of $N \times N$ matrices and contractive mappings Φ on a metric space.

Theorem 3. *Let $\mathcal{A} = \{\boldsymbol{A}_j : j \in \mathbb{Z}_n\}$ be a family of $N \times N$ matrices and $\Phi = \{\phi_j : j \in \mathbb{Z}_n\}$ a family of contractive mappings on a metric space which is of finite density and yields a locally connected invariant set with fix point intersections.*

A necessary and sufficient condition for the existence of a nontrivial continuous vector field $\boldsymbol{f} : \Omega \to \mathbb{R}^N$, *where* Ω *is the invariant set associated with* Φ, *is the existence of a subspace* $W \subseteq \mathbb{R}^N$ *and a set of nonzero vectors* $\{\boldsymbol{u}_j : j \in \mathbb{Z}_n\}$ *with the following properties:*

1. $\boldsymbol{A}_j^T \boldsymbol{u}_j = \boldsymbol{u}_j$, $j \in \mathbb{Z}_n$.
2. *If* $e, e' \in \mathbb{Z}_n^\infty$, $j, j' \in \mathbb{Z}_n$ *and* $\phi_e(x_j) = \phi_{e'}(x_{j'})$, *then*
$$\boldsymbol{A}_e^T \boldsymbol{u}_j = \boldsymbol{A}_{e'}^T \boldsymbol{u}_{j'}.$$
3. *There exists a subspace* $W \subseteq \mathbb{R}^N$ *such that for any* $j, k \in \mathbb{Z}_n$ *we have that*
$$\boldsymbol{u}_j - \boldsymbol{u}_k \in W, \quad \boldsymbol{A}_k^T \boldsymbol{u}_j - \boldsymbol{A}_j^T \boldsymbol{u}_k \in W.$$
and
$$V = U + W,$$
where $U = \operatorname{span}\{\boldsymbol{u}_j \ : \ j \in \mathbb{Z}_n\}$ *and* $V = \operatorname{span}\{\boldsymbol{A}_e^T \boldsymbol{u}_j \ : \ e \in \mathbb{Z}_n^\infty, \ j \in \mathbb{Z}_n\}$.
4. *The joint spectral radius* $\rho(\mathcal{A}, \Phi, W)$ *is less than one.*
Moreover, when 1)–4) *are satisfied, there is a* $\boldsymbol{y} \in \mathbb{R}^N$ *such that the refinable vector field* $\boldsymbol{f}$ *has the property that* $\boldsymbol{y}^T \boldsymbol{f} = 1$.

This is the main result of this paper whose proof follows by combining the results from the Secs. 2 and 3, cf. [1] for the special case $X = \mathbb{R}^s$, $\Omega = [0,1]^s$ and $\phi_e(x) := (x+e)/2$, $e \in \{0,1\}^s$, $x \in \mathbb{R}^s$. In the last section, we look at an example prompted by the fact that in general the *compatibility conditions* appearing in 2 are countably infinite in number, see also [1], although in the case considered in [7] there is only *one* compatibility condition.

5. The Square: An Example

When we choose our metric space to be $\mathbb{R}^2$ and the contractions to be
$$\phi_{\epsilon_1+2\epsilon_2}(x,y) = \Big(\frac{x+\epsilon_1}{2}, \frac{y+\epsilon_2}{2}\Big), \quad \epsilon_1, \epsilon_2 \in \{0,1\},$$
the corresponding invariant set is the unit square $\Omega = [0,1]^2$. In this case the refinement equations take the form
$$\begin{aligned} &\boldsymbol{A}_2^T\, \boldsymbol{f}(x,y) = \boldsymbol{f}\Big(\frac{x}{2}, \frac{y+1}{2}\Big), \quad \boldsymbol{A}_3^T\, \boldsymbol{f}(x,y) = \boldsymbol{f}\Big(\frac{x+1}{2}, \frac{y+1}{2}\Big), \\ &\boldsymbol{A}_0^T\, \boldsymbol{f}(x,y) = \boldsymbol{f}\Big(\frac{x}{2}, \frac{y}{2}\Big), \quad \boldsymbol{A}_1^T\, \boldsymbol{f}(x,y) = \boldsymbol{f}\Big(\frac{x+1}{2}, \frac{y}{2}\Big), \end{aligned} \tag{30}$$
where $\boldsymbol{A}_0, \ldots, \boldsymbol{A}_3$ are $N \times N$ matrices, see [1] for the case of $N \times N$ matrices on a cube. According to Theorem 1, there is an *infinite* number of compatibility conditions on these four matrices. However, when $N = 2$ we shall show below that *only four* such conditions are required. To this end, we need to examine the case of two matrices on the unit interval first. For that purpose we consider the one parameter family
$$\boldsymbol{G}_0(\tau) := \begin{bmatrix} 1 & 0 \\ 1-\tau & \tau \end{bmatrix}, \quad \boldsymbol{G}_1(\tau) := \begin{bmatrix} 1-\tau & \tau \\ 0 & 1 \end{bmatrix}, \quad \tau \in (0,1), \tag{31}$$

of matrices whose refinable function with respect to the two mappings $\phi_\epsilon(x) := \frac{x+\epsilon}{2}$, $\epsilon \in \{0,1\}$, plays a prominent role in [7]. We denote by $\boldsymbol{g}_\tau$ the refinable function with respect to these matrices.

Let $\boldsymbol{A}_\epsilon$, $\epsilon \in \{0,1\}$, be two 2×2 matrices with the following properties:

1. $\boldsymbol{A}_\epsilon \boldsymbol{1} = \boldsymbol{1}$, $\epsilon \in \{0,1\}$,
2. there exist two left eigenvectors $\boldsymbol{u}_\epsilon = \boldsymbol{A}_\epsilon^T \boldsymbol{u}_\epsilon$, $\epsilon \in \{0,1\}$ which satisfy the compatibility condition $\boldsymbol{A}_0^T \boldsymbol{u}_1 = \boldsymbol{A}_0^T \boldsymbol{u}_1$,
3. for $\epsilon \in \{0,1\}$
$$\left| (\boldsymbol{A}_\epsilon)_{11} - (\boldsymbol{A}_\epsilon)_{21} \right| < 1.$$

Any pair of matrices which satisfies the above conditions admits a refinable function. In fact, the above three properties characterize the convergence of the associated *matrix subdivision scheme* to a continuous function, see [7] for details. Our first result points out that all matrices of the above type can be obtained from the pair $\boldsymbol{G}_\epsilon(\tau)$, $\epsilon \in \{0,1\}$, by similarity.

Lemma 7. *If the two matrices $\boldsymbol{A}_0, \boldsymbol{A}_1 \in \mathbb{R}^2$ satisfy the above three properties then there exist $\tau \in (0,1)$ and a non–singular matrix $\boldsymbol{S} \in \mathbb{R}^2$ such that $\boldsymbol{S1} = \boldsymbol{1}$ and*
$$\boldsymbol{A}_\epsilon = \boldsymbol{S}^{-1} \boldsymbol{G}_\epsilon(\tau) \boldsymbol{S}, \quad \epsilon \in \{0,1\}. \tag{32}$$

Proof. We briefly outline the proof. The first step is to show that there is a 2×2 matrix $\boldsymbol{S}$ with $\boldsymbol{S1} = \boldsymbol{1}$ and scalars τ_0, τ_1 with $|\tau_0| < 1$ and $|1 - \tau_1| < 1$ such that
$$\boldsymbol{A}_\epsilon = \boldsymbol{S}^{-1} \boldsymbol{G}_\epsilon(\tau_\epsilon) \boldsymbol{S}, \quad \epsilon \in \{0,1\}. \tag{33}$$

Next, we use the compatibility conditions, condition 2) above and draw the conclusion that $\tau_0 = \tau_1$, thereby proving the lemma. ∎

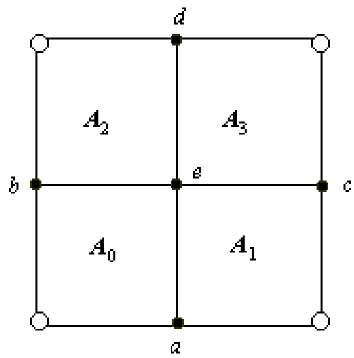

Fig. 1. Subdivision of the square.

Theorem 4. *Let the four matrices $\boldsymbol{A}_\epsilon$, $\epsilon \in \mathbb{Z}_4$, satisfy $\boldsymbol{A}_\epsilon \boldsymbol{1} = \boldsymbol{1}$, $\epsilon \in \mathbb{Z}_4$. There exists a nontrivial $(\Phi, \mathcal{A})$–refinable vector field $\boldsymbol{f}$ if and only if there exists a 2×2 invertible matrix $\boldsymbol{S}$ with $\boldsymbol{S1} = \boldsymbol{1}$ such that the four matrices $\boldsymbol{B}_\epsilon := \boldsymbol{S}^{-1} \boldsymbol{A}_\epsilon \boldsymbol{S}$, $\epsilon \in \mathbb{Z}_4$, have one of the following four forms:*

1.

$$\boldsymbol{B}_2 = \begin{bmatrix} 1 & 0 \\ 1-\tau & \tau \end{bmatrix}, \quad \boldsymbol{B}_3 = \begin{bmatrix} 1-\tau & \tau \\ 0 & 1 \end{bmatrix}, \qquad \tau \in (0,1). \tag{34}$$
$$\boldsymbol{B}_0 = \begin{bmatrix} 1 & 0 \\ 1-\tau & \tau \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} 1-\tau & \tau \\ 0 & 1 \end{bmatrix},$$

2.

$$\boldsymbol{B}_2 = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \boldsymbol{B}_3 = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \tag{35}$$
$$\boldsymbol{B}_0 = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}.$$

3.

$$\boldsymbol{B}_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{B}_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}, \tag{36}$$
$$\boldsymbol{B}_0 = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}.$$

4.

$$\boldsymbol{B}_2 = \begin{bmatrix} 1-\alpha & \alpha \\ \frac{1}{2}-\alpha & \frac{1}{2}+\alpha \end{bmatrix}, \quad \boldsymbol{B}_3 = \begin{bmatrix} \frac{1}{2}-\alpha & \frac{1}{2}+\alpha \\ -\alpha & 1+\alpha \end{bmatrix}, \qquad \alpha \in \mathbb{R}, \tag{37}$$
$$\boldsymbol{B}_0 = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \qquad \boldsymbol{B}_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

with corresponding refinable vectors

1.

$$\boldsymbol{f}(x,y) = \boldsymbol{g}_\tau(x), \quad x,y \in [0,1], \quad \tau \in (0,1),$$

2.

$$\boldsymbol{f}(x,y) = \begin{bmatrix} x \\ 1-x \end{bmatrix}, \quad x,y \in [0,1],$$

3.

$$\boldsymbol{f}(x,y) = \begin{bmatrix} \frac{1}{2}\left(x(2-y)+y(2-x)\right) \\ (1-x)(1-y) \end{bmatrix}, \quad x,y \in [0,1],$$

4.

$$\boldsymbol{f}(x,y) = \begin{bmatrix} 1-x-2\alpha y \\ x+2\alpha y \end{bmatrix}, \quad x,y \in [0,1], \quad \alpha \in \mathbb{R}.$$

Proof. That any of the above choices of $\boldsymbol{B}_\epsilon$, $\epsilon \in \mathbb{Z}_4$, can be verified easily by checking that vector fields given above are indeed refinable with respect to the associated matrices.

For the converse, we start by noting according to Lemma 7 that there exist a 2×2 invertible matrix $\boldsymbol{S}$ and $\tau \in (0,1)$ such that $\boldsymbol{S}\mathbf{1} = \mathbf{1}$ and $\boldsymbol{B}_\epsilon = \boldsymbol{S}^{-1}\boldsymbol{A}_\epsilon\boldsymbol{S}$, $\epsilon \in \{0,1\}$, for which $\boldsymbol{A}_\epsilon = \boldsymbol{B}_\epsilon(\tau)$. Next, we use the compatibility conditions at the points (b) and (c) in Fig. 1 to obtain that

$$\boldsymbol{B}_2 = \begin{bmatrix} 1 & 0 \\ b_2 & 1-b_2 \end{bmatrix} \quad \text{or} \quad \boldsymbol{B}_2 = \begin{bmatrix} b_2+1-\tau & \tau-b_2 \\ b_2 & 1-b_2 \end{bmatrix} \tag{38}$$

and

$$\boldsymbol{B}_3 = \begin{bmatrix} a_3 & 1-a_3 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \boldsymbol{B}_3 = \begin{bmatrix} b_3+\tau & 1-\tau-b_3 \\ b_3 & 1-b_3 \end{bmatrix}, \tag{39}$$

Choosing the two possibilities on the left hand side of (38) and (39), the compatibility conditions at (d) and (e) yield (34). Picking the left hand choice in (38) and the right hand one in (39) or vice versa, compatibility at (d) and (e) forces $\tau = 1/2$ and leads to (35) and (36), respectively. The situation is different for the matrices on the right hand side of (38) and (39) since compatibility at (d) is automatic then. Checking compatibility at (e), however, implies that $\tau = 1/2$ but leaves one parameter in the matrix free which leads to the representation (37). ∎

The proof of Theorem 4 shows that, although there is an infinite number of compatibility conditions in Theorem 2, it suffices to consider only the five compatibility conditions at the points (a)-(e) in Fig. 1.

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