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Properness and the Jacobian Conjecture in $\mathbb{R}^{2^*}$

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Abstract. A non-zero constant Jacobian polynomial map $F = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$ is a global diffeomorphism if the restrictions of F to every real level $P^{-1}(c)$, with $|c|$ large enough, is proper.

1. Introduction

Let $F(x, y) = (P(x, y), Q(x, y)) : k^2 \to k^2$ be a polynomial map, $k = \mathbb{R}; \mathbb{C}$. Let us denote by $J(F) := P_xQ_y - P_yQ_x$ the Jacobian of F. The famous Jacobian conjecture (see in [2] and [10]), posed by Keller [13] in 1939 and still open, asserts that F must have a polynomial inverse if its Jacobian $J(F)$ is a non-zero constant. If F has a polynomial inverse, of course, the levels of P are simply connected and the restrictions of F to every such level is a proper map. In the complex case it is well-known (see [3, 5, 7]) that for a non-zero constant Jacobian polynomial map F, the simply connectedness of a level L of P or the properness of the restriction of F to L, is sufficient for the existence of the polynomial inverse of F. These observations were deduced from Abhyankar-Moh-Suzuki theorem ([1, 16]) on embeddings of the line into the plane.

In this article we present some sufficient conditions, in term of connectedness, for non-zero constant Jacobian polynomial maps of $\mathbb{R}^2$ to be diffeomorphisms. Our results are the following.

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Theorem 1.1. *Suppose $F = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$ is a non-zero constant Jacobian polynomial map. If for sufficiently large $|c|$ the restriction of F on every level $P^{-1}(c)$ of P is proper, then F is a global diffeomorphism of $\mathbb{R}^2$.*

Theorem 1.2. *Suppose $F = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$ is a non-zero constant Jacobian polynomial map. If for sufficiently large $|c|$ the level $P^{-1}(c)$ is connected, then F is a global diffeomorphism of $\mathbb{R}^2$.*

Under the sufficient conditions of these theorems, it would be useful to check if, given a field k of characteristic zero, a bijective polynomial map of k^n with non-zero constant Jacobian has a polynomial inverse. This statement, which is part of the real plane Jacobian Problem, was asserted in [2, Theorem 1.2.1]. Nevertheless, its proof goes well only when k is algebraically closed (see in [4, 8]).

The proof of Theorem 1.1 and Theorem 1.2 presented in Sec. 3 is based on Lemma 2.3 which is a variant of the result in [6] related to the exceptional value set for non-zero constant Jacobian polynomial maps of $\mathbb{C}^2$. As discussed in Sec. 4, it is reasonable to ask here whether *Theorems* 1.1 *and* 1.2 *are still true for nonsingular polynomial maps of* $\mathbb{R}^2$. Notice that there are non-injective non-singular polynomial maps of $\mathbb{R}^2$. These very surprising examples were first constructed by Pinchuk [15] in 1993.

Theorems 1.1 and 1.2 are not valid for the analytic maps of $\mathbb{R}^2$. In Sec. 4, we shall construct a non-zero constant Jacobian analytic map $F = (P, Q): \mathbb{R}^2 \to \mathbb{R}^2$ which is non-injective, non-surjective, at most 2-to-1 and such that for all $|c|$ large enough, $P^{-1}(c)$ has exactly one connected component and also the restriction of F to each level $P^{-1}(c)$ is proper. Specially, this map is locally diffeomorphic and it preserves the area. It will be seen that this maps is a sort of "algebraic map".

2. Non-proper Value Set

Following [11], the *non-proper value set* A_h of a continuous map $h : \mathbb{R}^k \longrightarrow \mathbb{R}^k$ is the set of all values $a \in \mathbb{R}^k$ which have no neighborhood with compact inverse image under h. So, A_h is always a closed set and is a subset of the closure of the image of h. The non-proper value set A_h is a subset of the so-called *exceptional value set* E_h of h, which is defined to be the smallest subset of $\mathbb{R}^k$ such that $h : \mathbb{R}^k \setminus h^{-1}(E_h) \longrightarrow \mathbb{R}^k \setminus E_h$ gives an unbranched covering. When h is local homeomorphic and has finite fibers, the non-proper value set A_h coincides with E_h and is just the discontinue set of the integer-valued function $\#h^{-1}(a)$ defined on $\mathbb{R}^k$. If $A_h = \emptyset$, then h is a homeomorphism of $\mathbb{R}^k$.

We begin with a well-known fact on the geometry of non-proper value sets in the polynomial case.

Proposition 2.1. (see in [11]) *The non-proper value set A_h of a polynomial map $h = (p, q) : k^2 \longrightarrow k^2$, $k = \mathbb{R}, \mathbb{C}$, if not empty, is composed of the images of a finite number of non-constant polynomial maps from k into k^2.*

To see such description of A_h one can use a regular extension of F constructed by blowing up. We refer to [9, 14] for more details on this regular extension. In fact, in the complex case, one can regard h as a rational map from $\mathbb{CP}^2$ to $\mathbb{CP}\times\mathbb{CP}$, which may be undefined at a finite number of points in the line at infinity. Then, by blowing-up recursively, one can obtain a variety M containing $\mathbb{C}^2$ as an embedded subvariety and a regular extension $\bar{h}$ of h, $\bar{h} = (\bar{p}, \bar{q}) : M \longrightarrow \mathbb{CP}\times\mathbb{CP} \supset \mathbb{C}^2$. The components $\bar{p}$ and $\bar{q}$ of $\bar{h}$ are regular extensions of p and q, respectively. The exceptional curve $E := M - \mathbb{R}^2$ consists of a finitely many nonsingular rational components which meet to each other transversal and, at most, at double points. The graph associated with E is a tree. A component l of E is called a *dicritical component* of $\bar{h}$ if $\bar{h}(l)\cap\mathbb{C}^2 \neq \emptyset$. For such a dicritical component l there is a point $z_l \in l$ such that $\bar{h}(z_l) \notin \mathbb{C}^2$ and $\bar{h}(l\setminus\{z_l\}) \subset \mathbb{C}^2$. Then, one can see that

$$A_h = \bigcup_{l \text{ is a dicritical component of } \bar{h}} \bar{h}(l - \{z_l\}).$$

An analogous process can be done for the real case. We refer the readers to [11] for more details.

Polynomial maps defining A_h can be constructed by using Newton-Puiseux expansions, as done for the complex case in [6].

Lemma 2.2. *The non-proper value set A_h of a non-zero constant Jacobian polynomial map $F = (P, Q) : k^2 \longrightarrow k^2$, $k = \mathbb{R}; \mathbb{C}$, if not empty, must be composed of the images of a finite number of non-constant polynomial maps $f = (p, q)$ from k into k^2 with*

$$\frac{\deg p}{\deg q} = \frac{\deg P}{\deg Q}.$$

In particular, A_h cannot contain the image of a line (and a semi-line for the real case) under an automorphism of k^2.

The complex case of Lemma 2.2 was proved in [6, Theorem 4.4] by a Newton-Puiseux expansion approach. To give a proof for the real case we need the following elementary fact.

Regard the real affine space $\mathbb{R}^k$ as a subset of the complex affine space $\mathbb{C}^k$. Then, for a polynomial map $f = (f_1, f_2, \ldots, f_m) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ we can naturally define the extension $f_c := (f_1, f_2, \ldots, f_m) : \mathbb{C}^n \longrightarrow \mathbb{C}^m$.

Lemma 2.3. *Suppose $h : \mathbb{R}^2 \to \mathbb{R}^2$ and $\phi : \mathbb{R} \to \mathbb{R}^2$ are polynomial maps. Then,*

i) $A_h \subset A_{h_c}$;

ii) *If $\phi(\mathbb{R}) \subset A_h$, then $\phi_c(\mathbb{C}) \subset A_{h_c}$.*

Proof.

(i) follows from definition of a non-proper value set. If ϕ is a constant map,

(ii) is trivial. In the other case, the real curve $\phi(\mathbb{R})$ is one-dimensional and is contained in the intersection of the complex curve A_{h_c} and the irreducible curve

$\phi_c(\mathbb{C})$, by (i) and Proposition 2.1. So, $\phi_c(\mathbb{C})$ must be an irreducible component of the curve A_{h_c}. ■

Proof of the real case of Lemma 2.2. Given a non-zero constant Jacobian polynomial map $F = (P, Q) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Assume that the non-proper value set A_F is not empty. Then, by Proposition 2.1, A_F is composed of the images of some non-constant polynomial maps from $\mathbb{R}$ into $\mathbb{R}^2$. Let $\phi = (p, q) : \mathbb{R} \longrightarrow A_F \subset \mathbb{R}^2$ be such a parametrization map. By Lemma 2.3 the rational curve $\phi(\mathbb{C})$ is an irreducible component of the non-proper value A_{F_c} of F_c and ϕ_c is a parametrization map of A_{F_c}. Note that the curve $\phi(\mathbb{C}) \subset \mathbb{C}^2$ has only one irreducible branch at infinity. The fraction $\deg p/\deg q$ is equal to the first non-zero power appearing in Newton-Puiseux expansion at infinity of this branch, which does not depend on the choice of the parametrization map of $\phi(\mathbb{C})$. Then, by applying the complex case of the lemma to F_c, one can get all desired conclusions. ■

3. Proofs of Theorem 1.1 and Theorem 1.2.

We need the following lemma.

Lemma 3.1. *Let $F = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map. Assume that for $|c|$ large enough the restrictions of F on the levels $P^{-1}(c)$ of P are proper. Then, the non-proper value set A_F of F, if not empty, must be composed of some lines and semi-lines.*

Proof. Assume that for $|c| > R > 0$ the restrictions of F, to the levels $P^{-1}(c)$ of P are proper. From definitions we can easily see that if $L \subset \mathbb{R}^2$ is a line and the restriction of F on $F^{-1}(L)$ is proper, then $L \cap A_F = \emptyset$. This implies that the non-proper value set A_F of F must be contained in the set $\{(c, d) \in R^2 : |c| \leq R\}$. On the other hand, by Proposition 2.1, A_F is composed of the images of some non-constant polynomial maps $(p, q) : \mathbb{R} \longrightarrow \mathbb{R}^2$. Thus, the first component of such polynomial maps must be constant, and hence, A_F must consist of some lines and semi-lines parallel to the vertical axis. ■

Proof of Theorem 1.1. Let $F = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-zero constant Jacobian polynomial map and assume that, for every $|c| > R > 0$, the restrictions of F to the level $P^{-1}(c)$ is proper. Then, by Lemma 3.1 either $A_F = \emptyset$ or the non-proper value set A_F of F consists of some line and semi-line. The late is impossible by Lemma 2.2. Thus, $A_F = \emptyset$, and hence, the non-zero constant Jacobian map $F : \mathbb{R}^2 \to \mathbb{R}^2$ gives an unbranched covering. By the simple connectedness of $\mathbb{R}^2$, F must be a diffeomorphism of $\mathbb{R}^2$. ■

Proof of Theorem 1.2. Let $F = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-zero constant Jacobian polynomial map and assume that for $|c| > R > 0$ the levels $P^{-1}(c)$ are connected. Let $W := \{(c, d) \in \mathbb{R}^2 : |c| > R\}$. Since Q is monotone along each connected component of levels of P, F takes injectively $F^{-1}(W)$ into W. It follows from the definition that $E \cap A_F = \emptyset$. Therefore, for $|c| > R$ the restrictions of F on the levels $P^{-1}(c)$ of P are proper. Thus, by Theorem 1.1

the map F is a diffeomorphism of $\mathbb{R}^2$. ∎

4. Discussions and Examples

i) In order to understand the nature of the real Jacobian conjecture it is worth verifying this conjecture when the conditions on connectedness and properness in Theorem 1.1 and Theorem 1.2 hold only for one level of P.
ii) As shown in Lemma 3.1 the properness condition in Theorem 1.1 ensures that the non-proper value set A_F of F has a very special configuration: it is either empty or it must consist of some lines and semi-lines parallel to the vertical axis. It seems that Theorems 1.1 and 1.2 are still true for nonsingular polynomial maps of $\mathbb{R}^2$. However, the argument of this article, which is dominated by the Jacobian condition, cannot cover this situation.
iii) Theorem 1.1 and Theorem 1.2 are not valid for analytic maps of $\mathbb{R}^2$. More precisely, we shall construct a non-zero constant Jacobian analytic map $F = (P, Q)\colon \mathbb{R}^2 \to \mathbb{R}^2$ which is non-injective, non-surjective, at most 2-to-1 and such that for all $|c|$ large enough, $P^{-1}(c)$ has exactly one connected component and also the restriction of F to each level $P^{-1}(c)$ is proper.

First consider the map $F_1 = (P_1, Q_1)\colon (0,\infty)\times\mathbb{R} \to \mathbb{R}^2$ given by $P_1(x,y) = x(y^2-1)$ and $P_2(x,y) = xy(y^2-4)$. Then
(1) $\det(DF_1(x,y)) = x(y^4+y^2+4) > 0$ everywhere;
(2) if $c < 0$ then $P_1^{-1}(c)$ is a connected set which is the graph of the map $y \to x = c/(y^2-1)$ defined in $(-1,1)$;
(3) if $c > 0$ then $P_1^{-1}(c)$ has two connected components which are the graph of the maps $x \to y = \sqrt{(c+x)/x}$ and $x \to y = -\sqrt{(c+x)/x}$ defined in $(0,\infty)$;
(4) $P_1^{-1}(0)$ has two connected components: $\{y = 1\}$ and $\{y = -1\}$;
(5) F_1 is not injective because $F_1(1,2) = F_1(1,-2) = (3,0)$;
(6) F_1 is not surjective because $(0,0) \notin F_1((0,\infty)\times\mathbb{R})$;
(7) for all $c \in \mathbb{R}\setminus\{1\}$, F_1 restricted to each level $P_1^{-1}(c)$ is a proper map.

Now consider the analytic diffeomorphism $H_1\colon (0,\infty)\times\mathbb{R} \to (0,\infty)\times\mathbb{R}$ given by $H_1(x,y) = (\sqrt{2x}, h(y))$, where the diffeomorphism $h : \mathbb{R}\to\mathbb{R}$ is the solution of the differential equation:

$$h' = \frac{1}{h^4+h^2+4}, \quad h(0) = 0.$$

We may see that $h(y)$ satisfies the algebraic equation $(h(y))^5/5 + (h(y))^3/3 + 4(h(y)) = y$. Let $F_2 = F_1 \circ H_1$. We may check that $\det(DF_2) \equiv 1$.

As H_1 takes vertical lines onto vertical lines, there is a diffeomorphism $f\colon (0,\infty) \to \mathbb{R}$ such that $H_1(\{(x, f(x)) : x \in (0,\infty)\}$ is the connected component $\{(x, -\sqrt{(5+x)/x}) : x \in (0,\infty)\}$ of $P^{-1}(5)$. Define the area preserving analytic diffeomorphism $H_2\colon (0,\infty)\times\mathbb{R} \to (0,\infty)\times\mathbb{R}$ by $H_2(x,y) = (x, y+f(x))$. Observe that H_2 takes the positive first quadrant $\{(x,y)\in\mathbb{R}^2 : x > 0, y > 0\}$ onto the set $\{(x,y)\in\mathbb{R}^2 : x > 0, y > f(x)\}$ which in turn is taken by H_1 onto $\{(x,y)\in\mathbb{R}^2 : x > 0, y > -\sqrt{(5+x)/x}\}$.

We conclude that $F_3 = (P_3, Q_3) = F_1 \circ H_1 \circ H_2$, restricted to the first positive quadrant, has the following properties:
(1) $\det(DF_3(x, y)) = 1$ everywhere;
(2) if $c < 0$ then $P_3^{-1}(c)$ is a connected set;
(3) if $c \in (0, 5)$ then $P_1^{-1}(c)$ has two connected components and if $c \geq 5$, $P_3^{-1}(c)$ is a connected set;
(4) $P_1^{-1}(0)$ has two connected components;
(5) F_3 is non-injective and non-surjective;
(6) for all $c \in \mathbb{R} \setminus \{1\}$, F_3 restricted to each level $P_3^{-1}(c)$ is a proper map.

Let $H_3 : \mathbb{R}^2 \to (0, \infty) \times \mathbb{R}$ and $H_4 : \mathbb{R}^2 \to \mathbb{R} \times (0, \infty)$ be the following area preserving diffeomorphisms:

$$H_3(x, y) = \Big(x + \sqrt{x^2 + 4}, y\Big(\frac{\sqrt{x^2 + 4}}{x + \sqrt{x^2 + 4}}\Big)\Big),$$
$$H_4(x, y) = \Big(x\Big(\frac{\sqrt{y^2 + 4}}{y + \sqrt{y^2 + 4}}\Big), y + \sqrt{y^2 + 4}\Big).$$

Observe that the function $k(x) = x + \sqrt{x^2 + 4}$ satisfies the algebraic equation $(k(x))^2 - xk(x) = 4$. It can be seen that $H_3 \circ H_4$ takes $\mathbb{R}^2$ onto the first positive quadrant and that $F = (P, Q) = F_3 \circ H_3 \circ H_4$ is the example as required at the beginning of this section.

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