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On Complex Stability Radii for Implicit Discrete Time Systems

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Abstract. In this paper we study stability radii for implicit systems of linear difference equations. Both the cases when perturbations appear in either of two coefficient matrices are considered. Computable formulas of the complex stability radii are given. Beside some immediate conclusions concerning differences between explicit and implicit systems, a slight difference between the cases of differential and difference equations is also pointed out. In addition, we show an asymptotic relation between the stability radius for an implicit system of differential equations and the corresponding one for its discretized system obtained by the implicit Euler formula.

1. Introduction

In the last decade, considerable attention has been devoted to the study of robust stability for continuous and discrete time systems e.g., see [3–5, 7–9, 11–14]. Here, we consider the implicit system described by linear difference equations (the discrete time case)

$$Ax_{n+1} = Bx_n, \quad n = 0, 1, 2, ... \tag{1.1}$$

where $x_n, n = 0, 1, ...$, are complex or real vectors; A, B are square constant matrices of complex or real entries, the leading coefficient matrix A may be singular. Dynamical models of such a kind play an important role in a lot of application areas, e.g., population dynamics, biology, modeling of electrical networks, simulation of mechanical systems, etc. Furthermore, discretizations of an implicit system of differential equations

$$Ax'(\tau) = Bx(\tau) \tag{1.2}$$

lead to equations (1.1) as well. Following the concept of stability radii developed by Hinrichsen and Pritchard [7, 8], assuming the asymptotic stability of (1.1), we like to determine the infimums r_C and r_L of the norm of disturbances under which the perturbed systems

$$Ax_{n+1} = (B + E\Delta F)x_n \tag{1.3}$$

and

$$(A + E\Delta F)x_{n+1} = Bx_n \tag{1.4}$$

are no longer asymptotically stable, respectively. Here, Δ is an unknown disturbance matrix, E and F are matrices of appropriate size specifying the structure of the perturbation. The norm used here is a given matrix norm induced by a vector norm. In general, the Euclidean norm is used. If complex perturbations are allowed, the infimum values r_C and r_L are called the complex stability radii. In the case when only real perturbations are considered, the real stability radii are obtained. Problems of determining the stability radii for continuous time systems (1.2) are stated by analogy. The definition of the stability radius r_C can be considered as a natural extension of that for explicit systems. In the latter perturbed system (1.4), the perturbation affects the leading coefficient matrix of the system (1.1), only. In this paper, we focus on the complex stability radii for discrete time systems.

In the case of a nonsingular A, an implicit system of equations can be rewritten into the explicit form by multiplying both sides of the system with A^{-1}. Then, the problem of determining the stability radius r_C turns into the well-investigated problem with $A = I$. Results on explicit systems in both of the continuous and discrete time cases can be found in [7 - 9, 13]. Beside those papers, we mention some other related results on this field: a general computational formula for the real stability radius in [11], asymptotic behavior of the stability radius for a singularly perturbed system of differential equations in [5], the analysis of complex and real stability radii for linear delay systems in [14]. If matrix A is singular, in order to characterize the robust stability, investigation of a structure property, namely the index of matrix pencil $\{A, B\}$ is necessary. In a recent paper [4], Du and Lien have proposed a formula for computing the complex stability radius (with respect to perturbations in matrix B) for implicit systems of differential equations. In addition, some differences between the cases of ordinary differential equations (ODE-s) and differential-algebraic equations (DAE-s) have also been pointed out there. In the particular case, for index-1 differential-algebraic equations this formula of the stability radius was obtained in fact previously in [3, 12]. We have to underline that when an implicit system is considered even with a nonsingular A, it is worth giving a direct formula for the stability radius in order to avoid the inconvenient computation of A^{-1}. On the other hand, by multiplying with A^{-1}, the computation of the stability radius for explicit systems may become an ill-posed problem. For example, see the problem of computing the stability radius for a singularly perturbed system of differential equations in [5]. By using the formulas proposed in this paper, the computation of the stability radii leads to global optimization problems in general.

The paper is organized in the following way. In the next section we recall some preliminary results on the treatment of differential-algebraic equations and results on the stability radius obtained in [4]. Sec. 3 is concerned with the stability radius r_C for the discrete time system (1.1). Significant differences between the cases of nonsingular and singular systems are also pointed out there. In the next section, we proposed a formula for the stability radius r_L. As a conclusion, we know that the robust stability of implicit systems is more "sensitive" with respect to perturbations appearing in the leading coefficient matrix of the systems. The technique we apply here to prove the formulas for the stability radii of both kinds for (1.1) is quite similar to that used in [4]. However, there is a difference between the continuous and discrete time singular systems in the characterization of the robust stability, namely, if the stability radii are positive, there always exist destabilizing disturbances whose norm is exactly equal to the value of the stability radii. Finally, as an application, we establish an asymptotic relation between the stability radius for a continuous time system and the corresponding one for the discretized system obtained by the implicit Euler formula.

2. Preliminary Results for Continuous Time Systems

Consider the system of linear differential-algebraic equations

$$Ax'(\tau) = Bx(\tau) \tag{2.1}$$

where $x(.) : \mathbb{R} \to \mathbb{K}^m$ is a function, A and B are constant matrices in $\mathbb{K}^{m\times m}$, ($\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$), A is singular. The matrix pencil $\{A, B\}$ is supposed to be regular, i.e., there exists $\lambda \in \mathbb{C}$ such that $\det(\lambda A - B) \neq 0$. It is well-known that there exist nonsingular matrices W, T such that

$$A = W\begin{pmatrix} I_r & 0 \\ 0 & U \end{pmatrix} T^{-1}, \quad B = W\begin{pmatrix} B_1 & 0 \\ 0 & I_{m-r} \end{pmatrix} T^{-1} \tag{2.2}$$

where I_s denotes the identity matrix in $\mathbb{K}^{s\times s}$ and U is a matrix of nilpotency k having the Jordan block form, i.e., $U = \operatorname{diag}(J_1, ..., J_l)$ with

$$J_i = \begin{pmatrix} 0 & 1 & .. & 0 \\ .. & .. & .. & .. \\ 0 & .. & 0 & 1 \\ 0 & .. & 0 & 0 \end{pmatrix} \in \mathbb{K}^{p_i \times p_i}, \quad (i = 1, 2, .., l),$$

such that $\max_{1\le i\le l} p_i = k$. The above matrix decomposition is called the canonical Kronecker form of pencil $\{A, B\}$, see [2, 6]. If $U = 0$, then define $k = 1$. In the special case when A is nonsingular, we take $A = WIT^{-1}$ and $B = WB_1T^{-1}$ and define $k = 0$. If $\det(\lambda A - B)$ is identically constant, then the Kronecker form simplifies to $A = WUT^{-1}$, $B = WIT^{-1}$. The degree of nilpotency of U in the Kronecker form, namely the integer k, is the index of the matrix pencil $\{A, B\}$ and we write $\operatorname{index}\{A, B\} = k$.

We denote by $\sigma(A,B)$ the spectrum of the pencil $\{A,B\}$, i.e., the set of all λ satisfying the equation $\det(\lambda A - B) = 0$. If $A = I$, we simply write $\sigma(B)$ instead of $\sigma(I,B)$. It can be shown (see [6]) that system (2.1) is asymptotically stable if and only if all generalized eigenvalues of the pencil $\{A,B\}$ lie inside the left half of the complex plane, i.e., for each λ satisfying $\det(\lambda A - B) = 0$, its real part is negative. Let us suppose that system (2.1) is asymptotically stable and consider the disturbed system

$$Ax'(\tau) = (B + E\Delta F)x(\tau), \tag{2.3}$$

where $E \in \mathbb{K}^{m\times p}$, $F \in \mathbb{K}^{q\times m}$ are given matrices and $\Delta \in \mathbb{K}^{p\times q}$ is an uncertain disturbance. The matrix $E\Delta F$ is called a structured perturbation. Denote

$$\mathcal{V}_{\mathbb{K}} = \big\{\Delta \in \mathbb{K}^{p\times q}, \quad \text{system (2.3) is either unstable or irregular}\big\}.$$

Then, the stability radius of (2.1) can be defined by

$$d_K = \inf\big\{\|\Delta\|,\ \Delta \in \mathcal{V}_{\mathbb{K}}\big\},$$

where $\|.\|$ is an arbitrary matrix norm induced by a vector norm. In [4], a formula of the complex stability radius (the case $\mathbb{K} = \mathbb{C}$) has been proposed, namely

$$d_C = \big\{\sup_{\mathrm{Re}\,(s)\geq 0} \|G(s)\|\big\}^{-1} = \big\{\sup_{s\in i\mathbb{R}} \|G(s)\|\big\}^{-1}, \tag{2.4}$$

where $G(s) = F(sA - B)^{-1}E$ and $i\mathbb{R}$ denotes the pure imaginary co-ordinate line. Furthermore, some significant differences between the cases of singular and nonsingular A can be concluded immediately from formula (2.2) and the proof of (2.4), see [4]. In the next sections, we prove similar formulas for the complex stability radii of implicit difference equations.

3. The Stability Radius for Discrete Time Systems

In this section, we propose a formula of the stability radius r_C for the implicit system of difference equations

$$Ax_{n+1} = Bx_n, \quad n = 0, 1, \ldots \tag{3.1}$$

We suppose the same conditions on pair $\{A,\ B\}$ as in (2.1) and fix index$\{A,\ B\} = k$. We define the spectral radius for the matrix pencil $\{A,\ B\}$ by

$$\rho(A,B) = \max\big\{|\lambda|,\ \lambda \in \sigma(A,B)\big\}.$$

In the special case if $\sigma(A,\ B) = \emptyset$ we simply set $\rho(A,\ B) = 0$. Following the stability concept for differential-algebraic equations and difference equations, see [1, 6], we say that the trivial zero solution of the discrete time system (3.1) is asymptotically (and also exponentially) stable if there are a certain projection $P \in \mathcal{L}(\mathbb{K}^m)$ and a positive constant pair C, α such that the solution of initial value problem

$$\begin{cases} Ax_{n+1} = Bx_n, \quad n = 0, 1, \ldots \\ P(x_0 - \overline{x}_0) = 0 \end{cases}$$

exists uniquely and the estimate $\|x_n\| \le C\|P\overline{x}_0\|e^{-\alpha n}$, $n \ge 0$, holds. In case index $\{A, B\} = 1$, we choose $P = I - Q$ where Q is the projection onto $\mathrm{Ker}\,(A)$ along $S = \{z \in \mathbb{K}^m,\ Bz \in\ \mathrm{Im}\,A\}$, (see also recent results on linear implicit difference equations in [10]). If the zero solution of (3.1) is asymptotically stable, we then also say that system (3.1) is asymptotically stable. From the canonical Kronecker form, multiplying both sides of (3.1) by W^{-1}, we obtain the system

$$\begin{cases} y_{n+1} = B_1 y_n \\ U z_{n+1} = z_n \end{cases} \tag{3.2}$$

where $T^{-1}x_n = \begin{pmatrix} y_n \\ z_n \end{pmatrix}$, $y_n \in \mathbb{K}^r$, $z_n \in \mathbb{K}^{m-r}$. Since U is a nilpotent matrix of degree k, it is easy to deduce that the second equation $Uz_{n+1} = z_n$ has the unique solution $z_n \equiv 0$. Then, we arrive at

Lemma 1. *System* (3.1) *is asymptotically stable if and only if* $\rho(B_1)=\rho(A, B)<1$*, i.e., either* $|\lambda| < 1$ *for each* $\lambda \in \sigma(B_1) = \sigma(A, B)$ *or* $\sigma(A, B) = \emptyset$*.*

Proof. The result in the case $\sigma(A, B) \ne \emptyset$ is obtained directly by applying the well-known stability condition for the first equation of the equivalent system (3.2), see [1]. The case $\sigma(A, B) = \emptyset$ leads to the fact $\det(\lambda A - B)$ is identically a nonzero constant. Then, only the second equation appears in (3.2). Therefore, the trivial zero solution is the unique solution of system (3.1) and it can be considered asymptotically stable. ∎

From now on, we suppose that (3.1) is asymptotically stable and the same conditions hold for Δ, E, F as in (2.3). We obtain the result for the stability radius r_C defined in Sec. 1 as follows.

Theorem 1. *The complex stability radius* r_C *for system* (3.1) *is given by*

$$r_C = \Big\{ \sup_{|t|\ge 1} \|G(t)\| \Big\}^{-1} = \Big\{ \sup_{|t|\ge 1} \|F(tA - B)^{-1}E\| \Big\}^{-1} \tag{3.3}$$

Proof. First, we prove the inequality

$$r_C \ge \Big\{ \sup_{|t|\ge 1} \big\|G(t)\big\| \Big\}^{-1}. \tag{3.4}$$

Taking a disturbance matrix Δ which either destabilizes the system (3.1) or makes it irregular, then there are two cases:

a) The matrix pencil $\{A,\ B + E\Delta F\}$ is regular. It follows from Lemma 1 that there exists $\lambda \in \sigma(A, B + E\Delta F)$ such that $|\lambda| \ge 1$. Suppose that x is its corresponding eigenvector, that is $\lambda Ax = (B + E\Delta F)x$. It follows that $(\lambda A - B)x = E\Delta Fx$. Since the matrix pencil $\{A, B\}$ is stable, we can deduce that

$$x = (\lambda A - B)^{-1}E\Delta Fx \Rightarrow Fx = G(\lambda)\Delta Fx.$$

Hence the inequality

$$\|\Delta\| \ge \|G(\lambda)\|^{-1} \ge \Big\{ \sup_{|t|\ge 1} \|G(t)\| \Big\}^{-1} \tag{3.5}$$

holds for all disturbances Δ under which system (1.3) becomes unstable.
b) The matrix pencil $\{A,\, B+E\Delta F\}$ is irregular. Then, for any λ, $|\lambda|\geq 1$, there exists a vector $x\neq 0$ such that $\lambda Ax=(B+E\Delta F)x$. Then, by a similar argument, we can prove that (3.5) holds for all disturbances Δ under which the matrix pencil $\{A,\, B+E\Delta F\}$ becomes irregular.

We remind that the complex stability radius r_C is defined by

$$r_C=\inf\big\{\|\Delta\|;\text{ the pencil }\ \{A,B+E\Delta F\}\ \text{ is either unstable or irregular}\big\}.$$

Thus, the proof of (3.4) is complete.

Now, we prove the inverse inequality

$$r_C\leq\big\{\sup_{|t|\geq 1}\|G(t)\|\big\}^{-1}. \tag{3.6}$$

For any $\varepsilon>0$, there exists t_0 such that $|t_0|\geq 1$ and

$$\|G(t_0)\|^{-1}\leq\big\{\sup_{|t|\geq 1}\|G(t)\|\big\}^{-1}+\varepsilon.$$

We will construct a disturbance Δ such that $\|\Delta\|=\|G(t_0)\|^{-1}$. Since the norm used here is a matrix norm induced by a vector norm, there exists a vector $x\in\mathbb{C}^p$ such that $\|x\|=1$ and

$$\|G(t_0)x\|=\|G(t_0)\|.$$

It follows from a corollary of Hahn-Banach theorem that there exists a linear functional y^* defined in $\mathbb{C}^q$ such that $\|y^*\|=1$ and

$$y^*G(t_0)x=\|G(t_0)x\|=\|G(t_0)\|.$$

We set $\Delta=\|G(t_0)\|^{-1}xy^*$. Thus, one can easily check that $\Delta G(t_0)x=x$. Hence, we obtain $\|\Delta\|\geq\|G(t_0)\|^{-1}$. On the other hand, from its definition, we have immediately the inverse inequality $\|\Delta\|\leq\|G(t_0)\|^{-1}$. Therefore,

$$\|\Delta\|=\|G(t_0)\|^{-1}.$$

We now show that under the disturbance Δ defined above the disturbed system becomes either unstable or irregular. Indeed, after some elementary manipulation, the equality $\Delta G(t_0)x=x$ leads to $E\Delta Fu=(t_0A-B)u$, where $u=(t_0A-B)^{-1}Ex$. It is easy to check that $u\neq 0$ and then we conclude that $\det(t_0A-B-E\Delta F)=0$, i.e., either the pencil $\{A,\, B+E\Delta F\}$ is irregular or it is regular and $t_0\in\sigma(A,\, B+E\Delta F)$. In the latter case, since $|t_0|\geq 1$, the disturbed system

$$Ax_{n+1}=(B+E\Delta F)x_n$$

is unstable. Therefore, we obtain

$$r_C\leq\|\Delta\|=\|G(t_0)\|^{-1}\leq\big\{\sup_{|t|\geq 1}\|G(t)\|\big\}^{-1}+\varepsilon.$$

Because ε can be chosen arbitrarily, we have finished the proof of inequality (3.6).

Thus, by (3.4) and (3.6), the proof of Theorem 1 is complete. ■

We note that since $G(t)$ is analytic on the domain $D = \{t \in \mathbb{C},\, |t| \geq 1\}$, then by the maximum principle, $\|G(t)\|$ has its supremum at a finite point on the unit circle or at infinity. For a nonsingular matrix A, it is easy to verify that

$$\lim_{|t|\to\infty} \|G(t)\| = \lim_{|t|\to\infty} \left\|F(tA-B)^{-1}E\right\| = 0.$$

In case of a singular A, by using the canonical Kronecker form (2.2), we write

$$G(t) = F(tA-B)^{-1}E = FT \begin{pmatrix} (tI_r - B_1)^{-1} & 0 \\ 0 & -\sum_{i=0}^{k-1} (tU)^i \end{pmatrix} W^{-1}E, \quad (3.7)$$

and easily deduce that $\|G(t)\|$ tends to either a finite number or infinity when $|t| \to +\infty$.

Corollary 1. *The stability radius r_C can also be given by the formula*

$$r_C = \left[\max\left\{ \max_{|t|=1} \|G(t)\|,\ \lim_{t\to\infty} \|G(t)\| \right\}\right]^{-1}.$$

If the matrix A is nonsingular, the above formula is simplified to

$$r_C = \left\{ \max_{|t|=1} \|G(t)\| \right\}^{-1}.$$

We remark the latter result involves the corresponding formula for the stability radius in the case of explicit systems, that is, when $A = I$.

Corollary 2. *In the case $E = F = I$ (the unstructured perturbation) the stability radius $r_C > 0$ if and only if* index $\{A, B\} \leq 1$.

Now we turn to the question whether there exists a disturbance Δ destabilizing system (3.1) (or making it irregular) whose norm is r_C. From the proof of Theorem 1, we conclude that the answer depends on that $\|G(t)\|$ reaches its maximum at a finite point or at infinity. The existence of a "bad" disturbance is characterized as follows.

Theorem 2. *The following three statements are equivalent:*

a. *The stability radius r_C is positive.*
b. *The supremum of $\|G(t)\|$ is attained at a finite point (on the unit circle).*
c. *There exists a disturbance Δ such that $\|\Delta\| = r_C$ and the corresponding perturbed system is no longer stable.*

Proof. We will prove $a \Rightarrow b \Rightarrow c \Rightarrow a$.

a $\Rightarrow$ b. Suppose that $r_C > 0$. Then, by Corollary 1, we imply that $G_\infty = \lim_{|t|\to\infty} \|G(t)\| < \infty$. Using the transformation $s = 1/t$, the domain $|t| \geq 1$ is mapped onto the unit disk $|s| \leq 1$ on the complex plane and the unit circle is mapped onto itself. On the other hand, the function

$$\overline{G}(s) = \begin{cases} G(1/s), & 0 < s \le 1 \\ G_\infty, & s = 0 \end{cases}$$

is analytic on the unit disk. We note that, from (3.7), each entry of the matrix $G(t)$ is a linear combination of rational fractional functions and polynomials. Therefore, $\|\overline{G}(s)\|$ attains its maximum at a finite point on the unit circle, which immediately implies Statement *b*.

b $\Rightarrow$ c. Suppose that there exists a finite number $t_0, |t_0| = 1$ such that $\|G(t_0)\| = \sup_{|t|\ge 1} \|G(t)\|$. We construct a "bad" Δ as follows

$$\Delta = \|G(t_0)\|^{-1} x y^*,$$

where x, y^* are defined as in the proof of Theorem 1. It is naturally assured there that $\|\Delta\| = \|G(t_0)\|^{-1}$ and the system becomes either unstable or irregular under the disturbance Δ.

c $\Rightarrow$ a. This deduction is trivial. ∎

Moreover, it is obvious that $r_C = 0$ if and only if $\sup_{|t|\ge 1} \|G(t)\| = +\infty$. The latter can be attainable only at infinity and does not depend on the chosen norm. Also from the proof of Theorem 1, one can easily construct a "bad" disturbance whose norm is equal to an arbitrarily chosen positive ε. Finally, we remark that in the continuous time case, only the latter two statements are equivalent, see [4, Example 2].

Example 1. Consider system (3.1) with the following data

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 1/2 & -1 \\ -1 & 0 \end{pmatrix}.$$

It is easy to check that index$\{A, B\} = 1$, $\sigma(A, B) = \{-1/2\}$. With $E = F = I$, we have

$$G(t) = (tA - B)^{-1} = \begin{pmatrix} \frac{4t}{2t+1} & 1 \\ 1 & \frac{1}{2} \end{pmatrix}.$$

For sake of simplicity, we use the maximum vector norm and the induced matrix norm in this example and the others. We calculate

$$\sup_{|t|\ge 1} \|G(t)\|_\infty = \sup_{|t|\ge 1} \max \left\{ \left| \frac{4t}{2t+1} \right| + 1,\, 1 + \frac{1}{2} \right\} = 5.$$

This maximum is attainable at $t_0 = -1$. Then, we obtain $r_C = 1/5$. Based upon the proofs of Theorems 1, 2, it is easy to construct a "bad" disturbance

$$\Delta = \begin{pmatrix} 1/5 & 0 \\ 1/5 & 0 \end{pmatrix}.$$

This is indeed a destabilizing disturbance because a simple calculation shows $\sigma(A,\, B + \Delta) = \{-1\}$.

Example 2. Let the coefficient matrices of system (3.1) be given as follows:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 5/2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

It is easy to check that index$\{A, B\} = 2$, $\sigma(A, B) = \{1/2\}$. We have

$$(tA - B)^{-1} = \begin{pmatrix} \frac{2}{2t-1} & -\frac{2}{2t-1} & 0 \\ -\frac{2}{2t-1} & -\frac{2t-3}{2t-1} & -t+1 \\ 0 & 1 & t-2 \end{pmatrix}.$$

With the structured perturbation specified by

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F = I,$$

a simple calculation yields

$$\|G(t)\|_\infty = \left\|F(tA - B)^{-1}E\right\|_\infty = \max\left\{\frac{4}{|2t-1|}, \frac{2}{|2t-1|} + \left|\frac{2t-3}{2t-1}\right|, 1\right\}.$$

We obtain $\sup_{|t|\geq 1} \|G(t)\|_\infty = \|G(1)\|_\infty = 4$. Hence, $r_C = 1/4$. By some additional calculations, a "bad" disturbance is constructed as follows

$$\Delta = \begin{pmatrix} 1/4 & 0 & 0 \\ -1/4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

One can easily check that $\sigma(A,\, B + E\Delta F) = \{1\}$.

4. The Stability Radius with Respect to Perturbations in the Leading Coefficient Matrix

We aim now to give a formula for the stability radius r_L defined in Sec. 1, namely

$$r_L = \inf\left\{\|\Delta\|;\ \text{the pencil } \{A + E\Delta F, B\} \text{ is either unstable or irregular}\right\}$$

Analogously to Theorem 1, we formulate r_L as follows.

Theorem 3. *The complex stability radius r_L for system* (3.1) *is given by*

$$r_L = \left\{\sup_{|t|\geq 1} \|H(t)\|\right\}^{-1} \tag{4.1}$$

with $H(t) = Ft(tA - B)^{-1}E = F(A - B/t)^{-1}E$.

Proof. Suppose that Δ is a disturbance making the system either unstable or irregular. There exist then a number λ, $|\lambda| \geq 1$ and a vector $x \neq 0$ such that

$$\lambda(A + E\Delta F)x = Bx.$$

By a simple manipulation, we obtain

$$Fx = -F\Big(A - \frac{B}{\lambda}\Big)^{-1} E\Delta Fx = -H(\lambda)\Delta Fx.$$

Hence, we have an inequality similar to (3.5) which implies the analogy of (3.4) such as follows

$$r_L \geq \Big\{ \sup_{|t|\geq 1} \|H(t)\| \Big\}^{-1}.$$

The inverse inequality

$$r_L \leq \Big\{ \sup_{|t|\geq 1} \|H(t)\| \Big\}^{-1}$$

can be proven by a quite analogous procedure as in the proof of Theorem 1 (with only a change in the sign of Δ). The proof of (4.1) is complete. ∎

We obtain some conclusions similarly to Corollaries 1, 2 as well.

Corollary 3. *The stability radius r_L can also be given by the formula*

$$r_L = \Big[\max\Big\{ \max_{|t|=1} \|H(t)\|,\ \lim_{t\to\infty} \|H(t)\| \Big\}\Big]^{-1}.$$

Corollary 4. *In the case $E = F = I$ (the unstructured perturbation) then $r_L > 0$ if and only if* index$\{A, B\} = 0$, *i.e., A is a nonsingular matrix.*

The latter statement means exactly that in the case of a singular matrix A, the system (3.1) is very "sensitive" to perturbations affecting the leading coefficient matrix: an arbitrarily "small" disturbance can make the system become unstable or irregular. We may obtain a positive stability radius only in the case of structured perturbations.

In addition, we state that Theorem 2 remains valid with respect to the stability radius r_L. As a conclusion, in the case of a nonsingular matrix A, the formula of r_L is simplified to

$$r_L = \Big[\max_{|t|=1} \Big\| F\Big(A - \frac{B}{t}\Big)^{-1} E \Big\|\Big]^{-1}. \tag{4.2}$$

Example 3. We compute the stability radius r_L when the coefficient matrices are given as follows

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1/4 & -1/4 & 1/4 \\ -1/4 & -1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 \end{pmatrix}.$$

It is easy to check that index$\{A, B\} = 0$, $\sigma(A, B) = \{-1/2,\ 1/4,\ 1/2\}$. With the unstructured perturbation, i.e., $E = F = I$, we have

$$H(t) = Ft(tA - B)^{-1}E = \begin{pmatrix} \frac{2t}{2t-1} & -\frac{2t}{2t-1} & 0 \\ -\frac{2t}{2t-1} & \frac{8t^2}{4t^2-1} & -\frac{2t}{2t+1} \\ 0 & -\frac{2t}{2t+1} & \frac{16t^2+2t}{8t^2+2t-1} \end{pmatrix}.$$

We calculate

$$\sup_{|t|\geq 1} \|H(t)\|_\infty = \|H(1)\|_\infty = \frac{16}{3}.$$

Then, we obtain $r_C = 3/16$. Corresponding to the way constructing a "bad" disturbance as in the proof of Theorems 1 and 2, it is easy to obtain

$$\Delta = \begin{pmatrix} 0 & 3/16 & 0 \\ 0 & -3/16 & 0 \\ 0 & 3/16 & 0 \end{pmatrix}.$$

It is indeed a destabilizing disturbance since a simple calculation shows $\sigma(A + \Delta,\, B) = \{-1,\, 1/4,\, 1\}$.

Now, as an application of (4.1), we establish the asymptotic relation between the stability radius of a continuous time system described by (2.1) and that of the discretized system obtain by the implicit Euler formula as the stepsize h tends to zero. We assume again that the matrix pencil $\{A,\, B\}$ is regular. By applying the implicit Euler method which is a simple popular numerical method for solving IVP-s in ODE-s as well as in DAE-s to system (2.1), we obtain

$$A\frac{x_{n+1} - x_n}{h} = Bx_{n+1} \Leftrightarrow \Big(\frac{A}{h} - B\Big)x_{n+1} = \frac{A}{h}x_n. \tag{4.3}$$

Here $h > 0$ is the stepsize and x_n denotes the approximate solution to the exact solution $x(\tau)$ of (2.1) at the n-th node. The system of difference equations corresponding to the perturbed system (2.3) is as follows

$$\Big(\frac{A}{h} - B - E\Delta F\Big)x_{n+1} = \frac{A}{h}x_n. \tag{4.4}$$

Thus, we arrive at an implicit system with a structured perturbation in the leading coefficient matrix. First, we have the following statement.

Lemma 2. *If the continuous time system* (2.1) *is asymptotically stable, so is the discretized system* (4.3).

Proof. The assumption that (2.1) is asymptotically stable is equivalent to either the condition $\mathrm{Re}\,(z) < 0$ for all complex numbers z satisfying $\det(zA - B) = 0$, or $\sigma(A,\, B) = \emptyset$. In the former case, taking an arbitrary nonzero complex number λ, we have

$$\det\Big[\lambda\Big(\frac{A}{h} - B\Big) - \frac{A}{h}\Big] = \lambda^m \det\Big(\frac{\lambda - 1}{\lambda h}A - B\Big).$$

The above equality means that λ is a nonzero eigenvalue of the matrix pencil $\{(A/h) - B, A/h\}$ if and only if $z = \frac{\lambda-1}{\lambda h}$ is the corresponding one of the matrix pencil $\{A,\, B\}$. From the formula $\lambda = \frac{1}{1-zh}$, it is easy to see that the condition $\mathrm{Re}\,(z) < 0$ yields inequality $|\lambda| < 1$. Then, all eigenvalues of the matrix pencil $\{(A/h) - B, A/h\}$ have the modulus less than 1 (we note that $\lambda = 0$ is trivially an eigenvalue if A is singular). In case $\sigma(A,\, B) = \emptyset$, it is evident that the matrix pencil $\{(A/h) - B, A/h\}$ has at most the unique eigenvalue $\lambda = 0$. ∎

Remark 1. The property that solutions of the discretized system reflect the exponential decreasing property of solutions of the continuous time system is closely related to the stability concepts of numerical methods solving IVP-s for differential equations. In the numerical analysis literature [2, 6], the implicit Euler method is well-known to be A-stable. However, here we consider this property from another point of view, that is, before investigating the stability radius for a system, its asymptotic stability must be provided.

From the above lemma, we conclude that the stability radius for (4.3) is not less than that for (2.1). Denote the stability radius for (4.3) by $r_L(h)$ and that of (2.1) by d_C, respectively. We obtain now the asymptotic behavior for $r_L(h)$ when h tends to zero.

Theorem 4. *The stability radius with respect to perturbations in the leading matrix for the discretized system* (4.3) *monotone decreasingly tends to the stability radius of the original continuous time system* (2.1) *as the stepsize h tends to zero, that is*

$$\lim_{h\to+0} r_L(h) = d_C.$$

Proof. Since the leading coefficient matrix on the left hand side of (4.3) is nonsingular, applying formula (4.2) to the system of difference equations (4.3), we have

$$r_L(h) = \Big\{ \max_{|t|=1} \Big\| F\Big(\frac{A}{h} - B - \frac{A}{ht}\Big)^{-1} E \Big\| \Big\}^{-1} =$$

$$\Big\{ \max_{|t|=1} \Big\| F\Big(\frac{t-1}{ht} A - B\Big)^{-1} E \Big\| \Big\}^{-1} = \Big\{ \sup_{s\in S(1/h,1/h)} \|F(sA-B)^{-1}E\| \Big\}^{-1},$$

where $S(1/h, 1/h)$ denotes the disk having center at $1/h$ and radius $1/h$ on the complex plane. On the other hand, we recall the preliminary result (2.4)

$$d_C = \Big\{ \sup_{\mathrm{Re}(s)\geq 0} \|G(s)\| \Big\}^{-1} = \Big\{ \sup_{s\in i\mathbb{R}} \|G(s)\| \Big\}^{-1}.$$

The geometric interpretation of the result of Theorem 4 is as follows: when $h \to +0$, the disk $S(1/h, 1/h)$ grows and "almost fills out" the right hand half of the complex plane. First, it is easy to see that $r_L(h)$ is a monotone increasing function on the right hand side of zero since if $h_1 > h_2$ then $S(1/h_1, 1/h_1) \subset S(1/h_2, 1/h_2)$. On the other hand, function $r_L(h)$ has a lower bound, namely the stability radius d_C. Therefore, function $r_L(h)$ has the finite limit as h tends to zero and

$$\lim_{h\to+0} r_L(h) \geq d_C.$$

In order to prove the inverse inequality, it is sufficient to construct a sequence $\{h_j\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} h_j = 0$ and $\lim_{j\to\infty} r_L(h_j) \leq d_C$. We consider two cases:

a) If $\|G(s)\|$ attains its supremum (maximum) at a finite point $s_0 \in i\mathbb{R}$, we construct a complex number sequence $\{s_j\}_{j=1}^{\infty}$ as follows

$$\operatorname{Re}(s_j) = \frac{1}{h_j}\Big(1 - \frac{1}{\sqrt{|s_0|^2 h_j^2 + 1}}\Big), \ \operatorname{Im}(s_j) = \frac{|s_0|}{\sqrt{|s_0|^2 h_j^2 + 1}},$$

where $h_j = 1/j$. It is obvious that s_j is located on the boundary circle of the disk $S(1/h_j, 1/h_j)$ and $\lim_{j\to\infty} s_j = s_0$. Therefore,

$$\sup_{s\in S(1/h_j,1/h_j)} \|G(s)\| \geq \|G(s_0)\|.$$

On the other hand, because of continuity, we have

$$\lim_{j\to\infty} \|G(s_j)\| = \|G(s_0)\|.$$

Hence, we obtain

$$\lim_{h\to+0} r_L(h) = \lim_{j\to\infty} \big(\sup_{s\in S(1/h_j,1/h_j)} \|G(s)\|\big)^{-1} \leq \|G(s_0)\|^{-1} = d_C.$$

b) Otherwise, $\|G(s)\|$ attains its supremum at infinity and we can write

$$d_C = \Big\{\sup_{s\in i\mathbb{R}} \|G(s)\|\Big\}^{-1} = \lim_{s\to\infty} \|G(s)\|^{-1}.$$

We then put $s_j = 2/h_j$ with $h_j = 1/j$. Analogously to the previous case, it is clear that

$$\lim_{h\to+0} r_L(h) = \lim_{j\to\infty} \Big(\max_{s\in S(1/h_j,1/h_j)} \|G(s)\|\Big)^{-1} \leq \lim_{j\to\infty} \|G(s_j)\|^{-1} = d_C.$$

Right now, the proof of Theorem 4 is complete. ∎

Remark 2. By a completely similar procedure, one can prove that the stability radius r_C for system (4.3) tends to the stability radius d_C of (2.1) as the stepsize $h \to +0$, too. However, in our opinion, investigation of the relation between the stability radius r_L for (4.3) and the stability radius d_C for (2.1) seems to be more natural and reasonable.

5. Discussions

In this paper, we have proposed computable formulas of the complex stability radii for implicit systems of linear difference equations. The leading coefficient matrix of the systems may be singular. We consider both the cases when perturbations appear in either of two coefficient matrices. By investigating the formulas given here and the algebraic structure property, namely the index of the coefficient matrix pair, we have characterized some significant differences between the cases of explicit and implicit (especially singular) systems as well as a slight difference between the continuous and discrete time systems. In contrast to explicit systems, whose stability radii are always positive, implicit systems may have zero stability radii. In other words, the asymptotic stability property of implicit systems may be lost under an arbitrarily “small” perturbation. Furthermore, the stability properties of implicit systems are more “sensitive” with respect to perturbations occurring in the leading coefficient matrix than in the other. In addition, as an application, we have shown the asymptotic equality between the stability radius for an implicit continuous time system and the corresponding one for the discretized time system obtained by the implicit Euler method as the stepsize tends to zero.

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