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Short communication

The Cohomology of the Steenrod Algebra and Modular Representation Theory*

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1. Statement of Results

One of the most direct attempt in studying the cohomology of the Steenrod algebra by means of modular representations of the general linear groups was the surprising work [19] by Singer, which introduced a homomorphism, the so-called algebraic transfer, mapping from the coinvariants of certain representation of the general linear group to the cohomology of the Steenrod algebra.

Let $\mathbb{V}_k$ denote a k-dimensional $\mathbb{F}_2$-vector space, and $PH_*(B\mathbb{V}_k)$ the primitive subspace consisting of all elements in $H_*(B\mathbb{V}_k)$, which are annihilated by every positive-degree operation in the mod 2 Steenrod algebra, $\mathcal{A}$. Throughout the note, the homology is taken with coefficients in $\mathbb{F}_2$. The general linear group $GL_k := GL(\mathbb{V}_k)$ acts regularly on $\mathbb{V}_k$ and therefore on the homology and cohomology of $B\mathbb{V}_k$. Since the two actions of $\mathcal{A}$ and GL_k upon $H^*(B\mathbb{V}_k)$ commute with each other, there are inherited actions of GL_k on $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} H^*(B\mathbb{V}_k)$ and $PH_*(B\mathbb{V}_k)$. In [19], Singer defined the algebraic transfer

$$Tr_k : \mathbb{F}_2 \underset{GL_k}{\otimes} PH_d(B\mathbb{V}_k) \to \operatorname{Ext}_{\mathcal{A}}^{k,k+d}(\mathbb{F}_2, \mathbb{F}_2)$$

as an algebraic version of the geometrical transfer $tr_k : \pi_*^S((B\mathbb{V}_k)_+) \to \pi_*^S(S^0)$ to the stable homotopy groups of spheres.

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It has been proved that Tr_k is an isomorphism for $k = 1, 2$ by Singer [19] and for $k = 3$ by Boardman [2]. Among other things, these data together with the fact that $Tr = \oplus_k Tr_k$ is an algebra homomorphism (see [19]) show that Tr_k is highly nontrivial. Therefore, the algebraic transfer is expected to be a useful tool in the study of the mysterious cohomology of the Steenrod algebra, $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$.

Direct calculating the value of Tr_k on any non-zero element is difficult (see [19, 2, 8]). In this note, our main idea is to exploit the relationship between the algebraic transfer and the squaring operation Sq^0. It is well-known (see [13]) that there are squaring operations Sq^i $(i \geq 0)$ acting on the cohomology of the Steenrod algebra, which share most of the properties with Sq^i on the cohomology of spaces. However, Sq^0 is not the identity. On the other hand, there is an analogous squaring operation Sq^0, the Kameko one, acting on the domain of the algebraic transfer and commuting with the classical Sq^0 on $\mathrm{Ext}_{\mathcal{A}}^{k}(\mathbb{F}_2, \mathbb{F}_2)$ through the algebraic transfer.

The key point is that the behaviors of the two squaring operations do not agree in infinitely many certain degrees, called *k-spikes*. A k-spike degree is a number that can be written as $(2^{n_1} - 1) + \cdots + (2^{n_k} - 1)$, but can not be written as a sum of less than k terms of the form $(2^n - 1)$ (see [11]). The following result is originally due to Kameko [10]: If m is a k-spike, then

$$\widetilde{Sq}^0 : PH_*(B\mathbb{V}_k)_{\frac{m-k}{2}} \to PH_*(B\mathbb{V}_k)_m$$

is an isomorphism of GL_k-modules, where $\widetilde{Sq}^0$ is certain GL_k-homomorphism such that $Sq^0 = 1 \underset{GL_k}{\otimes} \widetilde{Sq}^0$.

We recognize two phenomena on the universality and the stability of k-spikes: First, if we start from any degree d that can be written as $(2^{n_1}-1)+\cdots+(2^{n_k}-1)$, and apply the function δ_k with $\delta_k(d) = 2d + k$ repeatedly at most $(k-1)$ times, then we get a k-spike; Secondly, k-spikes are mapped by δ_k to k-spikes. So, we have

Theorem 1.1. *Let d be an arbitrary non negative integer. Then*

$$(\widetilde{Sq}^0)^{i-k+2} : PH_*(B\mathbb{V}_k)_{2^{k-2}d+(2^{k-2}-1)k} \to PH_*(B\mathbb{V}_k)_{2^i d+(2^i-1)k}$$

is an isomorphism of GL_k-modules for every $i \geq k-2$.

From the result of Carlisle and Wood [5] on the boundedness conjecture, one can see that, for any degree d, there exists t such that

$$(\widetilde{Sq}^0)^{i-t} : PH_*(B\mathbb{V}_k)_{2^t d+(2^t-1)k} \to PH_*(B\mathbb{V}_k)_{2^i d+(2^i-1)k}$$

is an isomorphism of GL_k-modules for every $i \geq t$. However, this result does not confirm how large t should be. Theorem 1.1 shows that a rather small number $t = k-2$ commonly serves for every degree d. It will be pointed out in Remark 2 that $t = k-2$ is the minimum number for this purpose.

An inductive property of k-spikes, which will also play a key role in the note, is that if m is a k-spike, then $(2^n - 1 + m)$ is a $(k+1)$-spike for n big enough.

Two applications of the study will be exploited in this note. The first application is the following theorem, which is one of the note's main results.

Theorem 1.2. *Tr_k is not an isomorphism for $k \geq 5$. Furthermore, Tr_k is not an isomorphism in infinitely many degrees for each $k > 5$.*

In order to prove this theorem, using the notion of k-spike, we introduce the concept of critical element in $\mathrm{Ext}_{\mathcal{A}}^{k}(\mathbb{F}_2, \mathbb{F}_2)$ in such a way that if d is the stem of a critical element, then Tr_k is not an isomorphism either in degree d or in degree $2d + k$. Further, we show that if x is critical, then so is $h_n x$ for n big enough. Our inductive procedure starts with the initial critical element Ph_2 for $k = 5$.

Combining Theorem 1.2 and the results by Singer [19], Boardman [2] and Bruner–Hà–Hưng [4], we get

Corollary 1.3.

(i) *Tr_k is an isomorphism for $k = 1, 2$ and 3.*
(ii) *Tr_k is not an isomorphism for $k \geq 4$.*
(iii) *Tr_k is not an isomorphism in infinitely many degrees for $k = 4$ and $k > 5$.*

Remarkably, we do not know whether the algebraic transfer fails to be a monomorphism or fails to be an epimorphism for $k > 5$. Therefore, Singer's conjecture is still open.

Conjecture 1.4. [19] Tr_k is a monomorphism for every k.

The following theorem is related to this conjecture.

Theorem 1.5. *If Tr_ℓ detects a critical element, then it is not a monomorphism and further, Tr_k is not a monomorphism in infinitely many degrees for each $k > \ell$.*

A family $\{a_i |\ i \geq 0\}$ of elements in $\mathrm{Ext}_{\mathcal{A}}^{k}(\mathbb{F}_2, \mathbb{F}_2)$ (or in $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$) is called a *$Sq^0$-family* if $a_i = (Sq^0)^i(a_0)$ for every $i \geq 0$. The *root degree* of a_0 is the maximum non negative integer r such that $\mathrm{Stem}(a_0)$ can be written in the form $\mathrm{Stem}(a_0) = 2^r d + (2^r - 1)k$, for some non negative integer d.

The second application of our study is the following theorem, which is also one of the note's main results.

Theorem 1.6. *Let $\{a_i |\ i \geq 0\}$ be a Sq^0-family in $Ext_{\mathcal{A}}^{k}(\mathbb{F}_2, \mathbb{F}_2)$ and r the root degree of a_0. If Tr_k detects a_n for some $n \geq \max\{k - r - 2, 0\}$, then it detects a_i for every $i \geq n$ and detects a_j modulo $Ker(Sq^0)^{n-j}$ for $\max\{k - r - 2, 0\} \leq j < n$.*

A Sq^0-family is called *finite* if it has only finitely many non zero elements.

Corollary 1.7.

(i) *Every finite Sq^0-family in $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$ has at most $(k - 2)$ non zero elements.*

(ii) *If Tr_k is a monomorphism, then it does not detect any element of a finite Sq^0-family in $\mathrm{Ext}_{\mathcal{A}}^{k}(\mathbb{F}_2,\mathbb{F}_2)$ with at least $(k-1)$ non zero elements.*

The following is an application of Theorem 1.6 into the investigation of Tr_4.

Proposition 1.8. *Let $\{b_i|\ i\geq 0\}$ and $\{\overline{b}_i|\ i\geq 0\}$ be the Sq^0-families in $Ext_{\mathcal{A}}^4(\mathbb{F}_2,\mathbb{F}_2)$ with b_0 one of the usual five elements $d_0, e_0, p_0, D_3(0), p'_0$, and $\overline{b}_0$ one of the usual two elements f_0, g_1.*
(i) *If Tr_4 detects b_n for some $n\geq 1$, then it detects b_i for every $i\geq 1$.*
(ii) *If Tr_4 detects $\overline{b}_n$ for some $n\geq 0$, then it detects $\overline{b}_i$ for every $i\geq 0$.*

Based on this event, we prove the following theorem by showing that Tr_4 does not detect g_1, $D_3(1)$, p'_1.

Theorem 1.9. *Tr_4 does not detect any element in the three Sq^0-families $\{g_i|\ i\geq 1\}$, $\{D_3(i)|\ i\geq 0\}$ and $\{p'_i|\ i\geq 0\}$.*

This theorem gives a further negative information on Minami's [14] conjecture that the localization of the algebraic transfer given by inverting Sq^0 is an isomorphism. The first negative answer to this conjecture was given in Bruner–Hà–Hưng [4] by showing that the element in $(Sq^0)^{-1}\mathrm{Ext}_{\mathcal{A}}^4(\mathbb{F}_2,\mathbb{F}_2)$ represented by the family $\{g_i|\ i\geq 1\}$ is not detected by $(Sq^0)^{-1}Tr_4$. From Theorem 1.9, the two elements in $(Sq^0)^{-1}\mathrm{Ext}_{\mathcal{A}}^4(\mathbb{F}_2,\mathbb{F}_2)$ represented respectively by the two families $\{D_3(i)|\ i\geq 0\}$ and $\{p'_i|\ i\geq 0\}$ are also not detected by $(Sq^0)^{-1}Tr_4$.

The following theorem would complete our knowledge in Corollary 1.3 on whether Tr_5 is not an isomorphism in infinitely many degrees.

Theorem 1.10. *If $h_{n+1}g_n$ is non zero, then it is not detected by Tr_5.*

It has been claimed by Lin [12] that $h_{n+1}g_n$ is non zero for every $n\geq 1$.

2. Key Lemmas

Our starting point is Kameko's reduction [10]: If m is a *k-spike*, then

$$Sq^0: (\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k))_{\frac{m-k}{2}} \to (\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k))_m$$

is an isomorphism. This reduction is exploited in the proofs of our results, in which the following three lemmas play the key roles.

Lemma 2.1. *If m is a k-spike, then so is $(2m+k)$.*

Lemma 2.2. *If m is a k-spike, then (2^n-1+m) is a $(k+1)$-spike for every n with $2^n\geq m+k-1$.*

Let $\alpha(m)$ denote the number of ones in the dyadic expansion of m and δ_k the function given by $\delta_k(d)=2d+k$.

Lemma 2.3. *If d is a non negative integer with $\alpha(d+k) \leq k$, then $\delta_k^{k-1}(d) = 2^{k-1}d + (2^{k-1}-1)k$ is a k-spike.*

Remark 1.

(a) Lemma 2.3 cannot be improved in the meaning that the number $\delta_k^{k-2}(d) = 2^{k-2}d + (2^{k-2}-1)k$ is not a k-spike in general, for instance with $d = 2^t + 1 - k$.

(b) However, a number could be a k-spike although it is not of the form $\delta_k^{k-1}(d)$ for any non negative integer d. For instance, this is the case of the following numbers with $k = 4$:

$$\mathrm{Stem}(e_2) = 80, \quad \mathrm{Stem}(f_1) = 40, \quad \mathrm{Stem}(p_2) = 144,$$
$$\mathrm{Stem}(D_3(2)) = 256, \quad \mathrm{Stem}(p'_2) = 288,$$

where $e_2, f_1, p_2, D_3(2), p'_2$ are the usual elements in $\mathrm{Ext}^4_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$. This observation will be helpful in the proof of Proposition 1.8.

Remark 2. Let $k = 5$ and $d = 0$. As $\delta_5^{5-2}(0) = 35$, Theorem 1.1 claims that

$$(\widetilde{Sq}^0)^{i-3} : PH_*(B\mathbb{V}_5)_{35} \to PH_*(B\mathbb{V}_5)_{5(2^i-1)}$$

is an isomorphism of GL_5-modules for $i \geq 3$. In the final section we will see that

$$Sq^0 : \mathbb{F}_2 \underset{GL_5}{\otimes} PH_*(B\mathbb{V}_5)_{15} \to \mathbb{F}_2 \underset{GL_5}{\otimes} PH_*(B\mathbb{V}_5)_{35}$$

is not a monomorphism. This shows that Theorem 1.1 can not be improved in the meaning that $(k-2)$ is, in general, the minimum times that we must repeatedly apply Sq^0 to get into "the isomorphism region" of the iterated squaring operations.

3. Afterthoughts

Conjecture 3.1. Tr_4 is a monomorphism that detects all elements in $\mathrm{Ext}^4_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ except the ones in the three Sq^0-families $\{g_i |\ i \geq 1\}$, $\{D_3(i) |\ i \geq 0\}$ and $\{p'_i |\ i \geq 0\}$.

Which elements in $\mathrm{Ext}^5_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ are detected by Tr_5?

This question can partially be answered by using the fact that $Tr = \oplus_k Tr_k$ is an algebra homomorphism and the information on elements detected by Tr_k for $k \leq 4$. For instance, $h_3 D_3(0) = h_0 d_2$ (see [3]) is presumably detected by Tr_5, as h_0 is detected by Tr_1 and d_2 is expectedly detected by Tr_4 (see Conjecture 3.1).

Based on Theorem 1.6 and concrete calculations, the following conjecture presents some "new" families, which are expectedly detected by Tr_5.

Conjecture 3.2. Tr_5 detects every element in the Sq^0-families initiated by the classes $n, x, h_0 g_2, D_1, H_1, h_1 D_3(0), h_2 D_3(0), Q_3, h_4 D_3(0), h_6 g_1, h_0 g_3$ of stems 31, 37, 44, 52, 62, 62, 64, 67, 76, 83, 92 respectively.

Conjectures 3.2 and 3.1 together with the fact that $Tr = \oplus_k Tr_k$ is an algebra homomorphism predict that Tr_5 detects all Sq^0-families initiated by the classes of stems < 125, except possibly the three families, which are respectively initiated by Ph_1, Ph_2 and h_0p'. Since $Sq^0(Ph_1) = h_2g_1$, every element of the Sq^0-family initiated by Ph_1 is not detected by Tr_5 (see [19] for Ph_1 and Theorem 1.10 for $h_{n+1}g_n$). It has been known that Tr_5 does not detect the Sq^0-family of exactly one non zero element $\{Ph_2\}$. We have no prediction on whether the Sq^0-family initiated by h_0p' of stem 69 is detected or not.

Remark 3. We still do not know whether Tr_k fails to be a monomorphism or fails to be an epimorphism for $k > 5$. If Singer's Conjecture 1.4 that Tr_k is a monomorphism for every k is true, then the algebraic transfer does not detect the kernel of Sq^0 in k-spike degrees.

This leads us to the study of the kernel of Sq^0 in $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$. The map

$$\widetilde{Sq}^0 : PH_*(B\mathbb{V}_k) \to PH_*(B\mathbb{V}_k)$$

is obviously injective. One would expect that the Kameko map

$$Sq^0 = 1 \underset{GL_k}{\otimes} \widetilde{Sq}^0 : \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k) \to \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$$

is also a monomorphism. However, this is false. Indeed, $PH_*(B\mathbb{V}_5)$ has dimension 432 and 1117 in degrees 15 and 35 respectively, while $\mathbb{F}_2 \underset{GL_5}{\otimes} PH_*(B\mathbb{V}_5)$ has dimension 2 and 1 in degrees 15 and 35 respectively.

Combining these data with the fact that $\mathrm{Ext}_{\mathcal{A}}^{5,5+15}(\mathbb{F}_2, \mathbb{F}_2)=\mathrm{Span}\{h_0^4h_4, h_1d_0\}$, we get

Proposition 3.3.

(i) $\mathrm{Ker}(Sq^0) \cap (\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k))$ *is nonzero for* $k = 5$ *and has an infinite dimension for* $k > 5$.

(ii) Tr_k *detects a non zero element in the kernel of* Sq^0 *for* $k = 5$ *and infinitely many elements in this kernel for each* $k > 5$.

It has been known (see [19, 2]) that Sq^0 is injective on $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$ for $k \leq 3$.

Conjecture 3.4. Sq^0 is a monomorphism in positive degrees of $\mathbb{F}_2 \underset{GL_4}{\otimes} PH_*(B\mathbb{V}_4)$. In other words, Sq^0 is a monomorphism in positive degrees of $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$ if and only if $k \leq 4$.

Conjecture 3.5. (Sq^0 is eventually isomorphic on the Ext groups.)

Let $\mathrm{Im}(Sq^0)^i$ denote the image of $(Sq^0)^i$ on $\mathrm{Ext}_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2)$. There is a number t depending on k such that

$$(Sq^0)^{i-t} : \mathrm{Im}(Sq^0)^t \to \mathrm{Im}(Sq^0)^i$$

is an isomorphism for every $i > t$.

In other words, $\mathrm{Ker}(Sq^0)^i = \mathrm{Ker}(Sq^0)^t$ on $\mathrm{Ext}_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2)$ for every $i > t$. As a consequence, any finite Sq^0-family in $\mathrm{Ext}_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2)$ has at most t non zero elements.

Is the conjecture true for $t = k - 2$?

An observation on the known generators of the Ext groups supports the above conjecture with t much smaller than $k-2$. It also leads us to the question on whether Sq^0 is an isomorphism on $\mathrm{Im}(Sq^0)^t \subset \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$ for some $t < k - 2$. (This question has an affirmative answer for $t = k - 2$.)

The results of this note will be published in detail elsewhere.

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