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A Remark on Semigroup Crossed Products

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Abstract. We prove that the crossed product considered by Murphy is related to the semigroup crossed product studied by Stacey.

1. Introduction

Attempts to extend ideas of the established theory of crossed products by automorphism groups have motivated Murphy to study certain class of crossed products by semigroups in [5], and he developed some aspects of the theory. Subsequently Stacey introduced in [8] an explicit formulation of a semigroup crossed product based on universal property of covariant representations of the systems. A remark of Stacey's [8, p. 212] and Murphy's discussion in his introduction [5] suggest that their different notions of crossed product are unrelated. This paper shows that their suggestion is not true, we prove that each crossed product introduced by Murphy is isomorphic to the one given by Stacey.

Brief structure of this paper: in Sec. 2 we summarize known results on the general version of Stacey's crossed products, and we redefine the crossed product of Murphy so it can be conveniently seen that the fundamental difference between the two notions lies in the covariant condition; we then prove in Sec. 3 that the crossed product of Murphy is isomorphic to the one given by Stacey. There may be in other way to prove the theorem, but we give a direct proof. From this we derive [5, Theorem 3.1] as a corollary of our theorem.

2. Preliminaries

We start by recalling basic definitions and results on semigroup crossed products in which we will be interested.

A homomorphism ϕ from C^*-algebra A to a multiplier algebra $M(B)$ of

C^*-algebra B is *extendible* if there is an approximate identity (a_λ) for A and a projection p_ϕ in $M(B)$ such that $\phi(a_\lambda) \to p_\phi$ strictly in $M(B)$. It is proved in [1, Proposition 3.1.1] that a homomorphism $\phi : A \to M(B)$ is extendible precisely when there is a strictly continuous homomorphism $\bar{\phi} : M(A) \to M(B)$ such that $\bar{\phi}|_A = \phi$. This extendibility endomorphism is needed in our context because we are dealing with nonunital C^*-algebras.

Throughout this paper we let Γ be a totally ordered discrete abelian group with positive cone Γ^+. A *semigroup dynamical system* (A, Γ^+, α) is a system consisting of a C^*-algebra A and an action α of Γ^+ by extendible endomorphisms of A (meaning that each α_x is extendible). A *covariant representation* of (A, Γ^+, α) on a Hilbert space H (or in a C^*-algebra) is a pair (π, V), in which π is a nondegenerate representation of A on H and an isometric representation $V : \Gamma^+ \to$ Isometries (H) satisfying the covariance condition

$$\pi(\alpha_x(a)) = V_x \pi(a) V_x^* \text{ for } a \in A,\ x \in \Gamma^+.$$

A *crossed product* associated to (A, Γ^+, α) is a C^*-algebra B together with a nondegenerate homomorphism $i_A : A \to B$ and a semigroup homomorphism $i_{\Gamma^+} : \Gamma^+ \to$ Isometries $(M(B))$ satisfying:

1. $i_A(\alpha_x(a)) = i_{\Gamma^+}(x) i_A(a) i_{\Gamma^+}(x)^*$ for all $a \in A$, $x \in \Gamma^+$.
2. For any other covariant representation (π, V) of (A, Γ^+, α) on H there is a nondegenerate representation $\pi \times V$ of B on H such that $\pi \times V \circ i_A = \pi$ and $\overline{\pi \times V} \circ i_{\Gamma^+} = V$.
3. B is generated by $\{i_A(a) i_{\Gamma^+}(x) : a \in A, x \in \Gamma^+\}$.

If a semigroup dynamical system $(A, \Gamma^+\alpha)$ admits a nontrivial covariant representation, then there exists a crossed product for the system, which is unique up to isomorphism (see [1, 4] for detail proof). We denote it by $A \times_\alpha \Gamma^+$.

Remark 1. Stacey considered in [8] a dynamical system by a single endomorphism (i.e. $\Gamma^+ = \boldsymbol{N}$) of unital C^*-algebras. He pointed out that by adopting the abstract definition modelled on that of [7] for group, crossed product is the best approach to study semigroup crossed products. Extending his dilation ideas, we can view the semigroup crossed product as a full corner of the group crossed product. Dilate the action α of Γ^+ on A to an action α_∞ of the group Γ on a direct limit C^*-algebra A_∞, this is obtained by taking a direct limit over Γ^+, an arbitrary algebra A_x in the direct system is just a copy of A, and for $x \le y$ in Γ^+, the map $\alpha_x^y : A_x \to A_y$ is the endomorphism α_{y-x}. Using notation $\alpha^x : A_x \to A_\infty$ as the canonical homomorphism, then the action α_∞ is given by $(\alpha_\infty)_x(\alpha^y(a)) = \alpha^{y-x}(a)$ for $a \in A_x$ where x is fixed in Γ^+ and y is an element of Γ^+ such that $y \ge x$. While the generalization from $\mathbf{N}$ to Γ^+ is routine, dealing with nonunital algebras causes substantial technical problems, we have to show that all canonical homomorphisms α^x extend to homomorphisms between multiplier algebras in the compatible way. Then Theorem 5.1.9 [1] asserts that semigroup crossed product $A \times_\alpha \Gamma^+$ is isomorphic to the full corner

$$\overline{i_{A_\infty} \circ \alpha^0}(1_{M(A)})(A_\infty \times_{\alpha_\infty} \Gamma)\overline{i_{A_\infty} \circ \alpha^0}(1_{M(A)}).$$

Using this corner realization it was proved in [3, Theorem 2.2.1] that if the algebra is unital and the endomorphism α_x is injective for all x then the system has a nontrivial crossed product. It was then showed in [6, Theorem 4.5] that this is true for nonunital algebras.

2.1. Murphy's Crossed Product

Murphy considered in [5] a different type of covariance condition. His dynamical system is containing a C^*-algebra A, and an action α of Γ^+ by automorphisms on A. A covariant representation (or homomorphism) of this system is a pair (ϕ, W) in which ϕ is a representation of A (not necessarily nondegenerate) on H and an isometric representation $W : \Gamma^+ \to Isometries(H)$ satisfying his covariance:

$$\phi(\alpha_x(a))W_x = W_x\phi(a) \quad \text{for} \quad a \in A, x \in \Gamma^+.$$

Then the crossed product for his system (A, Γ^+, α) is the C^*-algebra B generated by his canonical covariant homomorphism (i_A, i_{Γ^+}) which is universal, in the sense that every covariant homomorphism (ϕ, W) from (A, Γ^+, α) corresponds to a representation $\phi \times W$ of B such that

$$\phi \times W(i_A(a)i_{\Gamma^+}(x)) = i_A(a)i_{\Gamma^+}(x) \quad \text{for} \quad a \in A, x \in \Gamma^+.$$

For our convenience, we write Murphy's crossed product as a triple (B, i_A, i_{Γ^+}) consisting of a C^*-algebra B, a nondegenerate homomorphism $i_A : A \to B$ and an isometric homomorphism $i_{\Gamma^+} : \Gamma^+ \to \text{Isometries}\,(H)$ satisfying
1. $i_A(\alpha_x(a))i_{\Gamma^+}(x) = i_{\Gamma^+}(x)i_A(a)$ for all $a \in A$ and $x \in \Gamma^+$.
2. For every Murphy covariant representation (ϕ, W) of (A, Γ^+, α) with nondegenerate ϕ, there is a nondegenerate representation $\phi \times W$ of B such that $\phi \times W \circ i_A = \phi$ and $\overline{\phi \times W} \circ i_{\Gamma^+} = W$.
3. B is generated by $\{i_A(A)i_{\Gamma^+}(\Gamma^+)\}$.

Our definition of Murphy crossed product does not make an essential difference to his theory because with the definition above, the crossed product automatically has Murphy's universal property. To see this, let (π, W) be a Murphy representation of (A, Γ^+, α) on H and suppose π is degenerate. Then π_{H_π} is a nondegenerate representation of A on the essential subspace $H_\pi = \overline{\text{span}}\{\pi(a)h : a \in A, h \in H\}$, and that $\pi_{H_\pi} \oplus 0$ is unitarily equivalent to π. Murphy covariance condition of (π, W) implies that the subspace H_π is invariant under each W_x, and therefore so is $H_\pi^\perp$. Consequently W is unitarily equivalent to $W|_{H_\pi} \oplus W|_{H_\pi^\perp}$. But the pair $(\pi|_{H_\pi}, W|_{H_\pi})$ is a our covariant representation of (A, Γ^+, α), so there is a nondegenerate representation $\pi|_{H_\pi} \times W|_{H_\pi}$ of B on H_π such that $\pi|_{H_\pi} \times W|_{H_\pi} \circ i_A = \pi|_{H_\pi}$ and $\pi|_{H_\pi} \times W|_{H_\pi} \circ i_{\Gamma^+} = W|_{H_\pi}$. Now take $\pi \times W$ to be $(\pi|_{H_\pi} \times W|_{H_\pi}) \oplus 0$. Then $\pi \times W$ is a representation of B such that $\pi \times W(i_A(a)i_{\Gamma^+}(x)) = \pi(a)W_x$ for all $a \in A$ and $x \in \Gamma^+$.

We shall use our definition for Murphy crossed product, and denote it by $A \times_\alpha^M \Gamma^+$.

Remark 2. Suppose we consider the dynamical system $(\mathbf{C}, \Gamma^+, \text{id})$ where $\mathbf{C}$ is just the C^*-algebra of complex numbers. Then every isometric representation V of Γ^+ gives Murphy covariant representation (π, V) of the system by taking π to be the unital representation $z \mapsto z1$ of $\mathbf{C}$, and therefore covariant representations of

Γ^+ are in one to one correspondence with isometric representations of Γ^+. Thus the crossed product $\mathbf{C} \times_{\mathrm{id}}^M \Gamma^+$ is the universal C^*-algebra $C^*(\Gamma^+)$ generated by semigroup of isometries, which is called the Toeplitz algebra. On the other hand, our crossed product $\mathbf{C} \times_{\mathrm{id}} \Gamma^+$ is the group crossed product $\mathbf{C} \times_{\mathrm{id}} \Gamma$ which is $C^*(\Gamma) \simeq C(\hat{\Gamma})$ because Γ is abelian.

3. Results

We consider a particular dynamical system consisting of the algebra B_{Γ^+}, that is the closed subspace of $l^\infty(\Gamma)$ spanned by the functions $\{1_x : x \in \Gamma^+\}$ where

$$1_x(y) = \begin{cases} 1 & \text{if } y \geq x \\ 0 & \text{otherwise.} \end{cases}$$

The action τ of Γ^+ on this algebra is obtained from the automorphism σ_x on $l^\infty(\Gamma)$ defined by $\sigma_x(f)(y) = f(y-x)$, which leaves B_{Γ^+} invariant because $\sigma_x(1_y) = 1_{x+y}$; hence σ restricts to an action τ of Γ^+ by endomorphisms on B_{Γ^+}. It is trivially extendible because B_{Γ^+} is unital, it contains namely 1_0.

Let (A, Γ^+, α) be a dynamical system of Murphy's type. The algebra B_{Γ^+} is a commutative algebra, hence it is nuclear. We therefore think of the tensor product $B_{\Gamma^+} \otimes A$ as the universal C^*-algebra for commuting pairs of representations of B_{Γ^+} and A. One can show (or apply Lemma 2.3 [4]) that there is an action $\tau \otimes \alpha$ of Γ^+ by extendible endomorphisms (and injective) on $B_{\Gamma^+} \otimes A$ such that $(\tau \otimes \alpha)_x(f \otimes a) = \tau_x(f) \otimes \alpha_x(a)$ for $f \in B_{\Gamma^+}$ and $a \in A$.

Theorem 3.1. *Suppose (A, Γ^+, α) is a dynamical system in which each α_x is an automorphism. Then $(B_{\Gamma^+} \otimes A) \times_{\tau\otimes\alpha} \Gamma^+$ is isomorphic to Murphy's crossed product $A \times_\alpha^M \Gamma^+$.*

Proof. With $j_{B_{\Gamma^+}\otimes A}$ and j_{Γ^+} denoting the canonical homomorphisms of $B_{\Gamma^+} \otimes A$ and Γ^+, respectively into the crossed product $(B_{\Gamma^+} \otimes A) \times_{\tau\otimes\alpha} \Gamma^+$, we let $i_A : A \to (B_{\Gamma^+} \otimes A) \times_{\tau\otimes\alpha} \Gamma^+$ be the map defined by

$$i_A(a) = j_{B_{\Gamma^+}\otimes A}(1_0 \otimes a),$$

and take $i_{\Gamma^+} = j_{\Gamma^+}$. Then we show that the triple $((B_{\Gamma^+} \otimes A \times_{\tau\otimes\alpha} \Gamma^+), i_A, i_{\Gamma^+})$ is Murphy crossed product for (A, Γ^+, α).

The homomorphism $j_{B_{\Gamma^+}\otimes A} : B_{\Gamma^+} \otimes A \to (B_{\Gamma^+} \otimes A) \times_{\tau\otimes\alpha} \Gamma^+$ is nondegenerate, hence it gives two commuting nondegenerate homomorphisms $j_{B_{\Gamma^+}} : B_{\Gamma^+} \to (B_{\Gamma^+} \otimes A) \times_{\tau\otimes\alpha} \Gamma^+$ and $j_A : A \to (B_{\Gamma^+} \otimes A) \times_{\tau\otimes\alpha} \Gamma^+$ such that $j_{B_{\Gamma^+}\otimes A}(f \otimes a) = j_{B_{\Gamma^+}}(f) j_A(a) = j_A(a) j_{B_{\Gamma^+}}(f)$ for all $f \in B_{\Gamma^+}$ and $a \in A$. So if (a_λ) is an approximate identity for A, then it follows that $i_A(a_\lambda) = j_A(a_\lambda)$, and so i_A is nondegenerate.

Next we show that the pair (i_A, i_{Γ^+}) satisfies Murphy's covariance. For this we firstly check that $(j_{B_{\Gamma^+}}, j_{\Gamma^+})$ is our covariant pair for $(B_{\Gamma^+}, \Gamma^+, \tau)$. For an approximate identity (a_λ) of A, $j_{B_{\Gamma^+}\otimes A}(\tau_x \otimes \alpha_x(f \otimes a_\lambda)) = j_{B_{\Gamma^+}}(\tau_x(f)) j_A(a_\lambda))$ converges strictly to $j_{B_{\Gamma^+}}(\tau_x(f))$ in the multiplier algebra $M((B_{\Gamma^+} \otimes A) \times_{\tau\otimes\alpha} \Gamma^+)$, and on the other hand

$$j_{B_{\Gamma^+}\otimes A}(\tau_x \otimes \alpha_x(f \otimes a_\lambda)) = j_{\Gamma^+}(x) j_{B_{\Gamma^+}}(f) j_A(a_\lambda) j_{\Gamma^+}(x)^*$$

converges strictly to $j_{\Gamma^+}(x)j_{B_{\Gamma^+}}(f)j_{\Gamma^+}(x)^*$. The uniqueness of limit implies that $j_{B_{\Gamma^+}}(\tau_x(f)) = j_{\Gamma^+}(x)j_{B_{\Gamma^+}}(f)j_{\Gamma^+}(x)^*$ for all $f \in B_{\Gamma^+}$ and $x \in \Gamma^+$. Then we deduce from computations below that (i_A, i_{Γ^+}) is Murphy covariant pair for (A, Γ^+, α):

$$\begin{aligned} i_A(\alpha_x(a))i_{\Gamma^+}(x) &= j_{B_{\Gamma^+}\otimes A}(1_0 \otimes \alpha_x(a))j_{\Gamma^+}(x) \\ &= j_A(\alpha_x(a))j_{\Gamma^+}(x) \\ &= j_A(\alpha_x(a))j_{\Gamma^+}(x)j_{\Gamma^+}(x)^* j_{\Gamma^+}(x) \\ &= j_A(\alpha_x(a))j_{B_{\Gamma^+}}(\tau_x(1_0))j_{\Gamma^+}(x) \\ &= j_{B_{\Gamma^+}\otimes A}(\tau_x \otimes \alpha_x(1_0 \otimes a))j_{\Gamma^+}(x) \\ &= j_{\Gamma^+}(x)j_{B_{\Gamma^+}\otimes A}(1_0 \otimes a)j_{\Gamma^+}(x)^* j_{\Gamma^+}(x) \\ &= j_{\Gamma^+}(x)j_A(a) \\ &= i_{\Gamma^+}(x)i_A(a). \end{aligned}$$

Suppose (ϕ, W) is Murphy covariant representation for (A, Γ^+, α) on H. We want to obtain a nondegenerate representation ψ of $(B_{\Gamma^+} \otimes A) \times_{\tau\otimes\alpha} \Gamma^+$ such that $\psi \circ i_A = \phi$ and $\bar{\psi} \circ i_{\Gamma^+} = W$. We now construct a covariant representation for $(B_{\Gamma^+} \otimes A, \Gamma^+, \tau \otimes \alpha)$. Proposition 2.2 (ii) of [2] allows us to have a unital representation π_W of B_{Γ^+} defined by $\pi_W(1_x) = W_x W_x^*$ for $x \in \Gamma^+$, and such that (π_W, W) is a covariant representation of $(B_{\Gamma^+}, \Gamma^+, \tau)$. Combining this with Murphy's covariance of (ϕ, W), we then have $\pi_W(1_x)\phi(a) = W_x W_x^* \phi(\alpha_x(\alpha_x^{-1}(a))) = W_x \phi(\alpha_x^{-1}(a))W_x^* = \phi(\alpha_x(\alpha_x^{-1}(a)))W_x W_x^* = \phi(a)\pi_W(1_x)$. Thus the unital representation π_W of B_{Γ^+} commutes with the nondegenerate representation ϕ of A. Consequently there is a nondegenerate representation $\pi_W \otimes \phi$ of $B_{\Gamma^+} \otimes A$ such that $\pi_W \otimes \phi(f \otimes a) = \pi_W(f)\phi(a) = \phi(a)\pi_W(f)$. Routine computations show that the pair $(\pi_W \otimes \phi, W)$ is a covariant representation for $(B_{\Gamma^+} \otimes A, \Gamma^+, \tau \otimes \alpha)$ on H. Now take ψ to be the nondegenerate representation $(\pi_W \otimes \phi) \times W$ of the crossed product $(B_{\Gamma^+} \otimes A) \times_{\tau\otimes\alpha} \Gamma^+$, and this ψ certainly satisfies the requirements $\psi \circ i_A = \phi$ and $\bar{\psi} \circ i_{\Gamma^+} = W$.

To complete the proof, we have to check that $(B_{\Gamma^+} \otimes A) \times_{\tau\otimes\alpha} \Gamma^+$ is generated by $\{i_A(a)i_{\Gamma^+}(x) : a \in A, x \in \Gamma^+\}$. But $(B_{\Gamma^+} \otimes A) \times_{\tau\otimes\alpha} \Gamma^+$ is generated by $\{j_{B_{\Gamma^+}\otimes A}(1_y \otimes a)j_{\Gamma^+}(x) : a \in A, x, y \in \Gamma^+\}$, so we have only to write $j_{B_{\Gamma^+}\otimes A}(1_y \otimes a)$ as an element of the first generators, which is immediate from the following computations: $j_{B_{\Gamma^+}\otimes A}(1_y \otimes a) = j_{B_{\Gamma^+}}(1_y)j_A(a) = j_{B_{\Gamma^+}}(\tau_y(1_0))j_{B_{\Gamma^+}\otimes A}(1_0 \otimes a) = j_{\Gamma^+}(y)j_{\Gamma^+}(y)^* i_A(a) = i_{\Gamma^+}(y)[i_A(a^*)i_{\Gamma^+}(y)]^* = [\overline{i_A}(1)i_{\Gamma^+}(y)][i_A(a^*)i_{\Gamma^+}(y)]^*$. ■

We now consider the dynamical system $(B_{\Gamma^+}, \Gamma^+, \tau)$, direct system of C^*-algebras induced by $(B_{\Gamma^+}, \Gamma^+, \tau)$ is determined by injective homomorphisms $\tau_x^y = \tau_{y-x}$ from $(B_{\Gamma^+})_x := B_{\Gamma^+}$ into $(B_{\Gamma^+})_y := B_{\Gamma^+}$ for $y \geq x$ in Γ^+. We claim that $B_\Gamma := \overline{\text{span}}\{1_x : x \in \Gamma\}$ is the direct limit $(B_{\Gamma^+})_\infty$; and dilation of the semigroup action τ is the group action τ of Γ by automorphisms on B_Γ. We use the same notation, because their formula are exactly the same, except one is acting as endomorphisms on B_{Γ^+} and the other is acting as automorphisms on B_Γ. To justify our claim, view each C^*-algebra $(B_{\Gamma^+})_x$ in the direct system as $B_x := \overline{\operatorname{span}}\{1_u : u \geq -x\}$ via the isomorphism $\phi_x(f)(t) = f(t+x)$. With inclusion map ι_{y-x} from B_x into B_y for $y \geq x$ in Γ^+, we have $\phi_y \circ \tau_{y-x} = \iota_{y-x} \circ \phi_x$ for all $y \geq x$ in Γ^+. Thus the direct limit $(B_{\Gamma^+})_\infty$ is isomorphic to the direct limit of $B_x \longrightarrow B_y \longrightarrow B_z \cdots$, which is B_Γ. Apply the construction of [3, Theorem 2.1] and [6, Theorem 4.5] to the action τ to obtain the action of the group Γ. We then deduce that (B_Γ, Γ, τ) is the automorphic dilation of $(B_{\Gamma^+}, \Gamma^+, \tau)$.

Similarly, we dilate Murphy dynamical system (A, Γ^+, α). Isomorphism α_x^{-1} for $x \in \Gamma^+$, satisfies $\alpha_x^{-1} \circ \alpha_{y-x} = \mathrm{id} \circ \alpha_x^{-1}$ for all $y \geq x$ in Γ^+. Therefore the direct limit A_∞ is A, and its automorphic dilation is (A, Γ, α).

Consequently the direct system $(B_{\Gamma^+} \otimes A)_x \longrightarrow (B_{\Gamma^+} \otimes A)_y \longrightarrow \cdots$ induced by the dynamical system $((B_{\Gamma^+} \otimes A), \Gamma^+, \tau \otimes \alpha)$ has direct limit $B_\Gamma \otimes A$, and $\tau \otimes \alpha$ is the dilated action of the group Γ by automorphisms of $B_\Gamma \otimes A$. Hence $((B_\Gamma \otimes A), \Gamma, \tau \otimes \alpha)$ is the automorphic dilation of semigroup dynamical system $((B_{\Gamma^+} \otimes A), \Gamma^+, \tau \otimes \alpha)$. Thus we have proved the following lemma.

Lemma 3.2. $((B_\Gamma \otimes A), \Gamma, \tau \otimes \alpha)$ *is the automorphic dilation of* $((B_{\Gamma^+} \otimes A), \Gamma^+, \tau \otimes \alpha)$.

Corollary 3.3. [5, Theorem 3.1] *If* (A, Γ^+, α) *is a dynamical system in which each* α_x *is an automorphism, then Murphy crossed product* $A \times_\alpha^M \Gamma^+$ *is isomorphic to* $p[(B_\Gamma \otimes A) \times_{\tau \otimes \alpha} \Gamma]p$ *where* p *is the projection* $\overline{\iota \otimes id}(1_0 \otimes 1_{M(A)})$ *in* $B_\Gamma \otimes A \times_{\tau \otimes \alpha} \Gamma$ *with* $\iota : B_{\Gamma^+} \to B_\Gamma$ *is the inclusion map.*

Proof. We know from Theorem 3.1 that $A \times_\alpha^M \Gamma^+$ is isomorphic to the crossed product $(B_{\Gamma^+} \otimes A) \times_{\tau \otimes \alpha} \Gamma^+$, and from Lemma 3.2 that the dilated system of $(B_{\Gamma^+} \otimes A, \Gamma^+, \tau \otimes \alpha)$ is $(B_\Gamma \otimes A, \Gamma, \tau \otimes \alpha)$. Therefore the semigroup crossed product $(B_{\Gamma^+} \otimes A) \times_{\tau \otimes \alpha} \Gamma^+$ is the full corner $p(B_\Gamma \otimes A \times_{\tau \otimes \alpha} \Gamma)p$ where p is $\overline{\iota \otimes \mathrm{id}}(1_0 \otimes 1_{M(A)})$ in $B_\Gamma \otimes A \times_{\tau \otimes \alpha} \Gamma$. ∎

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