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On Length of Generalized Fractions $\mathbf{1}/(x_1^n, \ldots, x_d^n, \mathbf{1})$

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Abstract. Let M be a finitely generated module over a Noetherian local ring $(R, \mathfrak{m})$ with $\dim M = d$. Let $(x_1, \ldots, x_d)$ be a system of parameters of M and n a positive integer. Consider the length of generalized fraction $1/(x_1^n, \ldots, x_d^n, 1)$ as a function in n. In this paper, we will give examples to show that, in general, the length of generalized fraction $1/(x_1^n, \ldots, x_d^n, 1)$ is not a polynomial for n large enough. Moreover we relate this problem with the study of the length of the ring $R/(x_1^n, \ldots, x_d^n, I)$, where $I \subset R$ is some ideal of R.

1. Introduction

In this paper, we always assume that $(R, \mathfrak{m})$ is a Noetherian local ring and M is a finitely generated R-module with $\dim M = d$. Sharp and Zakeri [7] introduced so-called modules of generalized fractions which generalizes the usual theory of localization of modules: For a positive integer k, Sharp and Zakeri [7] defined so-called *triangular subsets* of R^k, and given such a triangular subset U of R^k, they constructed an R-module $U^{-k}M$ called *the module of generalized fractions* of M with respect to U. Especially, the set

$$U(M)_{d+1} = \{(y_1, \ldots, y_d, 1) \in R^{d+1} : \text{ there exists } j \text{ with } 0 \leq j \leq d \text{ such that } (y_1, \ldots, y_j) \text{ form a subset of a s.o.p of } M \text{ and } y_{j+1} = \cdots = y_d = 1\}$$

is a triangular subset of R^{d+1}. Note that the top local cohomology modules $H_{\mathfrak{m}}^d(M)$ may be viewed as the module of generalized fractions of M with respect to $U(M)_{d+1}$. Moreover, for a system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of M, the submodules

$$M(1/(x_1^{n_1}, \ldots, x_d^{n_d}, 1)) = \{m/(x_1^{n_1}, \ldots, x_d^{n_d}, 1) : m \in M\}$$

of $U(M)_{d+1}^{-d-1}M$ is of finite length. Let

$$q_{\underline{x};M}(n) = \ell(M(1/(x_1^n, \ldots, x_d^n, 1))).$$

Following Sharp and Hamieh [6], $q_{\underline{x};M}(n)$ is called *the length of the generalized fraction* $1/(x_1^{n_1}, \ldots, x_d^{n_d}, 1)$. In [6, Question 1.2], Sharp and Hamieh asked that if there exists a polynomial $F(X_1, \ldots, X_d)$ in d variables with rational coefficients such that

$$q_{\underline{x};M}(\underline{n}) = F(n_1, \ldots, n_d)$$

for all $n_1, \ldots, n_d$ large enough. The answer to this question is positive when $\dim M \leq 2$ or M is generalized Cohen-Macaulay of any dimension (see [6]), but in general, the answer is negative. The counter-examples can be found in [3, Theorems 1.1, 1.2] for the case where M is of any dimension $d \geq 3$. However, when $n_1 = \cdots = n_d = n$, the functions $\ell(M(1/(x_1^n, \ldots, x_d^n, 1)))$ considered in [3, Theorem 1.1] are polynomials in n. Therefore, it is natural to ask the next question.

Question. Does there exist a polynomial $F(X)$ in one variable X such that $q_{\underline{x};M}(n) = F(n)$ for n large enough?

In this paper we will show by means of examples that in general $q_{\underline{x};M}(n)$ is not a polynomial for n large enough. The counter-example to this question can be constructed in the case where M is of any dimension $d \geq 3$.

Theorem 1.1. *Let $d \geq 3$ and $1 \leq r \leq d-2$ be integers. Let $S = K[x_1, \ldots, x_d, y]$, the polynomial ring in $d+1$ variables over a field K. Let $\mathfrak{m} = (x_1, \ldots, x_d, y)S$ and f be a polynomial in $\mathfrak{m} \cap K[x_1, \ldots, x_{d-1}]$. Let $s \geq 2$ be an integer and $\overline{S} = S/(y^s - f)S$. Let $R = \overline{S}_{\mathfrak{m}}$, the localization of $\overline{S}$ with respect to $\mathfrak{m}$ and $M = (x_{r+1}, \ldots, x_d)R$. Then $\underline{x} = (x_1, \ldots, x_{d-1}, x_d + y)$ is a system of parameters of M and*

$$q_{\underline{x};M}(n) = sn^d - \ell(\overline{S}/(x_1^n, \ldots, x_r^n, y^n, x_{r+1}, \ldots, x_d)\overline{S}).$$

In particular, if $f = x_1^s$ then

$$q_{\underline{x};M}(\underline{n}) = \begin{cases} sn^d - sn^r & \text{if } n \equiv 0 \pmod{s} \\ sn^d - sn^r + i(s-i)n^{r-1} & \text{if } n \equiv i \pmod{s}, \text{ for all } i = 1, \ldots, s-1, \end{cases}$$

for all $n \geq 1$. In this case, $q_{\underline{x};M}(n)$ is defined by $[s/2]+1$ polynomials, therefore it is not a polynomial for n large enough.

To have the similar results as in Theorem 1.1 for rings, we need the concept of principle of idealizations, which was introduced by Nagata [5, p. 2]. We make the

Cartesian product $R \times M$ into a commutative ring with respect to componentwise addition and multiplication defined by $(r, m).(r', m') = (rr', rm' + r'm)$. We call this the *idealization* of M (over R) and denote it by $R \ltimes M$. The idealization $R \ltimes M$ is Noetherian local ring with identity $(1, 0)$, its maximal ideal is $\mathfrak{m} \times M$ and its Krull dimension is $\dim R$.

Corollary 1.2. *Let R and M be as in Theorem 1.1 and $R \ltimes M$ the idealization of M. Then $\underline{(x, 0)} = ((x_1, 0), \ldots, (x_{d-1}, 0), (x_d + y, 0))$ is a system of parameters of $R \ltimes M$ and*

$$q_{\underline{(x,0)};R\ltimes M}(n) = 2sn^d - \ell(\overline{S}/(x_1^n, \ldots, x_r^n, y^n, x_{r+1}, \ldots, x_d)\overline{S}).$$

In particular, if $f = x_1^s$ then

$$q_{\underline{(x,0)};R\ltimes M}(n) = \begin{cases} 2sn^d - sn^r & \text{if } n \equiv 0 \pmod{s} \\ 2sn^d - sn^r + i(s-i)n^{r-1} & \text{if } n \equiv i \pmod{s},\ i = 1, \ldots, s-1, \end{cases}$$

for all $n \geq 1$. In this case, $q_{\underline{(x,0)};R\ltimes M}(n)$ is defined by $[s/2] + 1$ polynomials, therefore it is not a polynomial for n large enough.

2. Proof of Theorem 1.1

In this section, we always assume that $d \geq 3$, $1 \leq r \leq d-2$ and $s \geq 2$ are integers. Firstly, we need the following lemmas

Lemma 2.1. [2, Lemma 2.3] *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. For a s.o.p $\underline{x} = (x_1, \ldots, x_d)$ of M, set*

$$Q(\underline{x}; M) = \bigcup_{t>0} (x_1^{t+1}, \ldots, x_d^{t+1})M :_M x_1^t \ldots x_d^t.$$

Then we have

$$M/Q(\underline{x}; M) \cong M(1/(x_1, \ldots, x_d, 1)).$$

Lemma 2.2. [3, Lemma 2.6] *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module with $\dim M = \dim R = d$. Denote by $R \ltimes M$ the idealization of M. Let $\underline{x} = (x_1, \ldots, x_d)$ be a s.o.p of R. Then $\underline{(x, 0)} = ((x_1, 0), \ldots, (x_d, 0))$ is a s.o.p of $R \ltimes M$ and*

$$\ell\big(R \ltimes M/Q(\underline{(x, 0)}; R \ltimes M)\big) = \ell(R/Q(\underline{x}; R)) + \ell(M/Q(\underline{x}; M)),$$

where $Q(\underline{x}; R), Q(\underline{x}; M)$ and $Q(\underline{(x, 0)}; R \ltimes M)$ are defined as in Lemma 2.1.

The proof of the next lemma uses some knowledge from Groebner basis [1].

Lemma 2.3. *Let $S = K[x_1, \ldots, x_d, y]$, the polynomial ring in $d+1$ variables over a field K, $\mathfrak{m} = (x_1, \ldots, x_d, y)S$ and $f \in \mathfrak{m} \cap K[x_1, \ldots, x_d]$ a polynomial of S. Then for any integers $t, n \geq 1$ we have*

$$(x_1^{nt+n}, \ldots, x_d^{nt+n}, y^s - f)S :_S x_1^{nt} \ldots x_d^{nt} = (x_1^n, \ldots, x_d^n, y^s - f)S.$$

Proof. Set
$$\mathfrak{a} = (x_1^{nt+n}, \dots, x_d^{nt+n}, y^s - f)S :_S x_1^{nt} \dots x_d^{nt}$$
and
$$\mathfrak{b} = (x_1^n, \dots, x_d^n, y^s - f)S.$$
Since $(x_1^{nt+n}, \dots, x_d^{nt+n})S$ is a monomial ideal, we have

$$\big((x_1^{nt+n}, \dots, x_d^{nt+n})S :_S x_1^{nt} \dots x_d^{nt}\big) + (y^s - f)S = (x_1^n, \dots, x_d^n, y^s - f)S.$$

Therefore $\mathfrak{b} \subseteq \mathfrak{a}$. Now we consider the lexicographic term order, where $y > x_1 > x_2 > \dots > x_d$. With this term order, we claim that $\mathrm{in}(\mathfrak{a}) = \mathrm{in}(\mathfrak{b})$. In fact, let g be an arbitrary polynomial in $\mathfrak{a}$. Then we have

$$\mathrm{in}(x_1^{nt} \dots x_d^{nt} g) = x_1^{nt} \dots x_d^{nt} \mathrm{in}(g) \in \mathrm{in}((x_1^{nt+n}, \dots, x_d^{nt+n}, y^s - f)S).$$

By Buchberger's criterion, we can easily check that $x_1^{nt+n}, \dots, x_d^{nt+n}, y^s - f$ is a Groebner basis of $(x_1^{nt+n}, \dots, x_d^{nt+n}, y^s - f)S$. So, we have
$$\mathrm{in}((x_1^{nt+n}, \dots, x_d^{nt+n}, y^s - f)S) = (x_1^{nt+n}, \dots, x_d^{nt+n}, y^s)S.$$
These imply that
$$x_1^{nt} \dots x_d^{nt} \mathrm{in}(g) \in (x_1^{nt+n}, \dots, x_d^{nt+n}, y^s)S.$$
Therefore,

$$\mathrm{in}(g) \in (x_1^{nt+n}, \dots, x_d^{nt+n}, y^s)S :_S x_1^{nt} \dots x_d^{nt} = (x_1^n, \dots, x_d^n, y^s)S.$$

Note that $\mathrm{in}(\mathfrak{b}) = (x_1^n, \dots, x_d^n, y^s)S$. Therefore $\mathrm{in}(g) \in \mathrm{in}(\mathfrak{b})$. This implies that $\mathrm{in}(\mathfrak{a}) = \mathrm{in}(\mathfrak{b})$ and the claim is proved. Since $\mathrm{in}(\mathfrak{a}) = \mathrm{in}(\mathfrak{b})$ and $\mathfrak{b} \subseteq \mathfrak{a}$, it follows that $x_1^n, \dots, x_d^n, y^s - f$ is also a Groebner basis of $\mathfrak{a}$. Thus,
$$\mathfrak{a} = (x_1^n, \dots, x_d^n, y^s - f)S = \mathfrak{b}.$$
■

Lemma 2.4. *Let S and f be as in Lemma 2.3. Let $\overline{S} = S/(y^s - f)S$. Then for any integers $t \geq sd$ and any $n \geq 1$ we have*

$$(x_1^{nt+n}, \dots, x_d^{nt+n})(x_{r+1}, \dots, x_{d-1}, x_d - y)\overline{S} :_{\overline{S}} x_1^{nt} \dots x_d^{nt} = (x_1^n \dots, x_d^n)\overline{S}.$$

Proof. Set
$$\mathfrak{a} = (x_1^{nt+n}, \dots, x_d^{nt+n})(x_{r+1}, \dots, x_{d-1}, x_d - y)\overline{S} :_{\overline{S}} x_1^{nt} \dots x_d^{nt}.$$
It is clear that $\mathfrak{a} \subseteq (x_1^{nt+n}, \dots, x_d^{nt+n})\overline{S} :_{\overline{S}} x_1^{nt} \dots x_d^{nt}$. Moreover, we have

$$\begin{aligned}
&(x_1^{nt+n}, \dots, x_d^{nt+n})\overline{S} :_{\overline{S}} x_1^{nt} \dots x_d^{nt} \\
&= \big((x_1^{nt+n}, \dots, x_d^{nt+n}, y^s - f)S :_S x_1^{nt} \dots x_d^{nt}\big)/(y^s - f)S \\
&= (x_1^n, \dots, x_d^n, y^s - f)S/(y^s - f)S = (x_1^n, \dots, x_d^n)\overline{S}
\end{aligned}$$

by Lemma 2.3. Therefore $\mathfrak{a} \subseteq (x_1^n, \dots, x_d^n)\overline{S}$. Conversely, since $1 \leq r \leq d-2$, it is easy to check that $(x_1^n, \dots, x_{d-2}^n, x_d^n)\overline{S} \subseteq \mathfrak{a}$. So, we need only to prove $x_{d-1}^n \overline{S} \subseteq \mathfrak{a}$. Let

$$\mathfrak{b} = (x_1^{nt+n}, \dots, x_d^{nt+n})(x_{r+1}, \dots, x_{d-1}, x_d - y)\overline{S}.$$

We have

$$x_{d-1}^{nt+n} x_1^{nt} \dots x_{d-2}^{nt} x_d^{nt} = x_{d-1}^{nt+n}(x_d - y) x_1^{nt} \dots x_{d-2}^{nt} x_d^{nt-1} + y x_{d-1}^{nt+n} x_1^{nt} \dots x_{d-2}^{nt} x_d^{nt-1}.$$

Hence

$$x_{d-1}^{nt+n} x_1^{nt} \dots x_{d-2}^{nt} x_d^{nt} \overline{S} \subseteq \mathfrak{b} \text{ if and only if } y x_{d-1}^{nt+n} x_1^{nt} \dots x_{d-2}^{nt} x_d^{nt-1} \overline{S} \subseteq \mathfrak{b}.$$

By the same arguments, after nt steps we get

$$x_{d-1}^{nt+n} x_1^{nt} \dots x_{d-2}^{nt} x_d^{nt} \overline{S} \subseteq \mathfrak{b} \text{ if and only if } y^{nt} x_{d-1}^{nt+n} x_1^{nt} \dots x_{d-2}^{nt} x_d^{nt-1} \overline{S} \subseteq \mathfrak{b}.$$

Since $t \geq sd$ and $f \in \mathfrak{m} \cap K[x_1, \dots, x_d]$ with notice that $y^s \overline{S} = f \overline{S}$, we obtain

$$y^{nt} x_{d-1}^{nt+n} x_1^{nt} \dots x_{d-2}^{nt} x_d^{nt-1} \overline{S} \subseteq f^{nd} x_{d-1}^{nt+n} x_1^{nt} \dots x_{d-2}^{nt} x_d^{nt-1} \overline{S} \subseteq \mathfrak{b}.$$

Therefore $x_{d-1}^n \overline{S} \subseteq \mathfrak{a}$. ■

Lemma 2.5. *Let $S, f, \overline{S}$ be as in Lemma* 2.4. *Then for any positive integers $t \geq sd$ and $n \geq 1$ we have*

$$\begin{aligned}(x_1^{nt+n}, \dots, x_{d-1}^{nt+n}, (x_d + y)^{nt+n})(x_{r+1}, \dots, x_d)\overline{S} :_{\overline{S}} x_1^{nt} \dots x_{d-1}^{nt}(x_d + y)^{nt} \\ = (x_1^n \dots, x_{d-1}^n, (x_d + y)^n)\overline{S}.\end{aligned}$$

Proof. Let

$$\mathfrak{a} = (x_1^{nt+n}, \dots, x_{d-1}^{nt+n}, (x_d + y)^{nt+n})(x_{r+1}, \dots, x_d)\overline{S} :_{\overline{S}} x_1^{nt} \dots x_{d-1}^{nt}(x_d + y)^{nt}$$

and

$$\mathfrak{b} = (x_1^n, \dots, x_{d-1}^n, (x_d + y)^n)\overline{S}.$$

Since $1 \leq r \leq d - 2$, we can check that $(x_1^n, \dots, x_{d-2}^n, (x_d + y)^n)\overline{S} \subseteq \mathfrak{a}$. So, we need only to prove $x_{d-1}^n \overline{S} \subseteq \mathfrak{a}$. Note that there exists a polynomial g such that

$$x_{d-1}^{nt+n} x_1^{nt} \dots x_{d-2}^{nt}(x_d + y)^{nt} = x_d x_{d-1}^{nt+n} x_1^{nt} \dots x_{d-2}^{nt} g + y^{nt} x_{d-1}^{nt+n} x_1^{nt} \dots x_{d-2}^{nt}.$$

So, similarly to the proof of Lemma 2.4, we have $x_{d-1}^n \overline{S} \subseteq \mathfrak{a}$ since both ideals $x_d x_{d-1}^{nt+n} x_1^{nt} \dots x_{d-2}^{nt} g \overline{S}$ and $y^{nt} x_{d-1}^{nt+n} x_1^{nt} \dots x_{d-2}^{nt} \overline{S}$ are contained in

$$(x_1^{nt+n}, \dots, x_{d-1}^{nt+n}, (x_d + y)^{nt+n})(x_{r+1}, \dots, x_d)\overline{S}.$$

Conversely, let $g(x_1, \dots, x_d, y)$ be a polynomial such that $g(x_1, \dots, x_d, y)\overline{S} \subseteq \mathfrak{a}$. By replacing $x_1 = x_1; \dots; x_d = x_d - y; y = y$, we have

$$g(x_1, \dots, x_d - y, y)\overline{S} \subseteq (x_1^{nt+n}, \dots, x_d^{nt+n})(x_{r+1}, \dots, x_{d-1}, x_d - y)\overline{S} :_{\overline{S}} x_1^{nt} \dots x_d^{nt}.$$

Therefore $g(x_1,\dots,x_d-y,y)\overline{S}\subseteq(x_1^n,\dots,x_d^n)\overline{S}$ by Lemma 2.4. Now by replacing $x_1=x_1;x_2=x_2;\dots;x_d=x_d+y,y=y$, we have

$$g(x_1,\dots,x_d,y)\overline{S}\subseteq(x_1^n,\dots,x_{d-1}^n,(x_d+y)^n)\overline{S}=\mathfrak{b}.$$

■

Lemma 2.6. *Let $r\geq 1, s\geq 2$ be integers, $T=K[x_1,\dots,x_r,y]$, the polynomial ring in $r+1$ variables over a field K and $\overline{T}=T/(y^s-x_1^s)T$. Then we have*

$$\ell(\overline{T}/(x_1^n,\dots,x_r^n,y^n)\overline{T})=\begin{cases} sn^r & \text{if } n\equiv 0 \pmod s\\ sn^r-i(s-i)n^{r-1} & \text{if } n\equiv i \pmod s,\ i=1,\dots,s-1,\end{cases}$$

for all $n\geq 1$.

Proof. Let

$$\mathfrak{a}=(x_1^n,\dots,x_r^n,y^n,y^s-x_1^s)T.$$

Suppose that $n=sl+i$, where $0\leq i<s$. Then we can check that

$$\mathfrak{a}=(x_1^n,\dots,x_r^n,y^s-x_1^s,y^ix_1^{sl})T.$$

Consider the reverse lexicographic term order with $y>x_1>\dots>x_d$. It is immediate to check that $x_1^n,\dots,x_r^n,y^s-x_1^s,y^ix_1^{sl}$ is a Groebner basis of $\mathfrak{a}$ and hence

$$\operatorname{in}(\mathfrak{a})=(x_1^n,\dots,x_r^n,y^s,y^ix_1^{sl})T.$$

It should be mentioned that the Hilbert functions with respect to $\mathfrak{a}$ and $\operatorname{in}(\mathfrak{a})$ are the same. Therefore the claim follows. ■

Proof of Theorem 1.1. Since $f\in\mathfrak{m}\cap k[x_1,\dots,x_{d-1}]$, $(x_1,\dots,x_{d-1},x_d+y)$ is a s.o.p of M. By the flatness of the natural homomorphism $\overline{S}\longrightarrow\overline{S}_{\mathfrak{m}}$, we get by [4, (3.H)] and by Lemma 2.5 that

$$Q(x_1^n,\dots,x_{d-1}^n,(x_d+y)^n;M)=(x_{r+1},\dots,x_d)\overline{S}_{\mathfrak{m}}\cap(x_1^n,\dots,x_{d-1}^n,(x_d+y)^n)\overline{S}_{\mathfrak{m}},$$

where $Q(x_1^n,\dots,x_{d-1}^n;M)$ is defined similarly as in Lemma 2.1. It follows from this relation and Lemma 2.1 that

$$\begin{aligned}q_{\underline{x};M}(n)&=\ell\big((x_{r+1},\dots,x_d)\overline{S}_{\mathfrak{m}}/(x_{r+1},\dots,x_d)\overline{S}_{\mathfrak{m}}\cap(x_1^n,\dots,x_{d-1}^n,(x_d+y)^n)\overline{S}_{\mathfrak{m}}\big)\\&=\ell\big((x_1^n,\dots,x_r^n,y^n,x_{r+1},\dots,x_d)\overline{S}_{\mathfrak{m}}/(x_1^n,\dots,x_{d-1}^n,(x_d+y)^n)\overline{S}_{\mathfrak{m}}\big).\end{aligned}$$

Note that $\overline{S}_{\mathfrak{m}}$ is Cohen–Macaulay. Therefore, we have

$$\begin{aligned}&q_{\underline{x};M}(n)=\ell(\overline{S}_{\mathfrak{m}}/((x_1^n,\dots,x_{d-1}^n,(x_d+y)^n)\overline{S}_{\mathfrak{m}})\\&-\ell(\overline{S}_{\mathfrak{m}}/(x_1^n,\dots,x_r^n,y^n,x_{r+1},\dots,x_d)\overline{S}_{\mathfrak{m}})\\&=e(x_1^n,\dots,x_{d-1}^n,(x_d+y)^n;\overline{S}_{\mathfrak{m}})-\ell(\overline{S}_{\mathfrak{m}}/(x_1^n,\dots,x_r^n,y^n,x_{r+1},\dots,x_d)\overline{S}_{\mathfrak{m}})\\&=sn^d-\ell(\overline{S}_{\mathfrak{m}}/(x_1^n,\dots,x_r^n,y^n,x_{r+1},\dots,x_d)\overline{S}_{\mathfrak{m}}).\end{aligned}$$

The case $f=x_1^s$ follows by Lemma 2.6. ■

Proof of Corollary 1.2. Since R is Cohen–Macaulay, $x_1^n,\dots,x_{d-1}^n,(x_d+y)^n$ is a regular R-sequence. It follows that

$$Q(x_1^n,\dots,x_{d-1}^n,(x_d+y)^n;R)=(x_1^n,\dots,x_{d-1}^n,(x_d+y)^n)R.$$

Therefore, by Lemma 2.1 we have

$$q_{\underline{x};R}(n)=\ell\big(\overline{S}_{\mathfrak{m}}/(x_1^n,\dots,x_{d-1}^n,(x_d+y)^n)\overline{S}_{\mathfrak{m}}\big)=sn^d.$$

Now the assertion follows by Lemma 2.1, Lemma 2.2 and Theorem 1.1. ∎

Remark. In our examples, although the lengths of generalized fractions are not always polynomials for large n, it is still very nice since they are defined by finitely many polynomials. Moreover, if we choose $f=y+x_1+x_2$ or $f=y-x_1^{a_1}x_2^{a_2}\dots x_r^{a_r}$ with $a_1,a_2,\dots,a_r$ positive integers then, by using Groebner basis, we can show that the length of the generalized fractions in Theorem 1.1 is also defined by finitely many polynomials. Therefore we close this paper by the following question.

Question. Let $(R,\mathfrak{m})$ be a Noetherian local ring, M a finitely generated R-module with $\dim M=d$. Under which condition on a s.o.p $(x_1,\dots,x_d)$ of M, the length of generalized fraction $1/(x_1^n,\dots,x_d^n,1)$ is defined by finitely many polynomials for n large enough?

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