Vietnam Journal of Mathematics **31**:4 (2003) 411–419

Vietnam Journal
of
MATHEMATICS

Remarks on Some Fuchsian Groups

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Received July 22, 2002
Revised January 13, 2003

Abstract. In this paper, we introduce some Fuchsian groups and we calculate parabolic class numbers of those groups.

1. Introduction

Let m be a square free positive integer and let $H(\sqrt{m})$ denote the discrete subgroup of $PSL(2,\mathbb{R})$, which consists of all those mappings of the form

i) $T(z) = \dfrac{az + b\sqrt{m}}{c\sqrt{m}z + d}$, $a, b, c, d \in \mathbb{Z}$, $ad - bcm = 1$,

ii) $T(z) = \dfrac{a\sqrt{m}z + b}{cz + d\sqrt{m}}$, $a, b, c, d \in \mathbb{Z}$, $adm - bc = 1$.

When $m = 2$ or $m = 3$, the resulting groups are the Hecke groups generated by the elements $z \to z + \sqrt{m}$ and $z \to -1/z$ [3, 4]. Let $H_0^m(n)$ be defined as follows

$$H_0^m(n) = \big\{T \in H(\sqrt{m}) \,:\, c \equiv 0(\operatorname{mod} n)\big\}.$$

Then $H_0^m(n)$ is a subgroup of $H(\sqrt{m})$ and $H_0^m(n)$ turn out to be the congruence subgroups of $H(\sqrt{m})$ associated to the ideal generated by $\tau = x + y\sqrt{m} \in \mathbb{Z}[\sqrt{m}]$ when $m = 2, 3$ (see [9]).

In [4], we dealt with $H(\sqrt{m})$ and we calculated parabolic class number of $H_0^m(n)$ in case m is a prime number. Now let $H^m(\sqrt{m})$ be the subgroup of $H(\sqrt{m})$, which is defined as follows

$$H^m(\sqrt{m}) = \big\{T \in H(\sqrt{m}) : a \equiv d \equiv 0(\operatorname{mod} m) \text{ or } b \equiv c \equiv 0(\operatorname{mod} m)\big\}$$

and let $K_0^m(n) = H^m(\sqrt{m}) \cap H_0^m(n)$. In this study we calculate parabolic class number of $K_0^m(n)$ in case m is a prime number and m is prime to n.

Let Λ be a discrete subgroup of $PSL(2,\mathbb{R})$. The parabolic subgroups of Λ are defined to be those non-identity cyclic subgroups $C \subset \Lambda$ which consist of parabolic elements (together with the identity) and which are maximal with respect to this property. The parabolic class number s of Λ is the number of conjugacy classes of parabolic subgroups of Λ (see [2]).

By a Fuchsian group Λ we will mean a finitely generated discrete subgroup of $PSL(2,\mathbb{R})$ the group of conformal homeomorphisms of the upper-half plane. The most general presentation for Λ is

Generators

$$a_1, b_1, \dots, a_g, b_g \quad \text{(hyperbolic)},$$
$$x_1, x_2, \dots, x_r \quad \text{(elliptic)},$$
$$p_1, p_2, \dots, p_s \quad \text{(parabolic)}.$$

Relations

$$x_1^{m_1} = x_2^{m_2} = \dots\dots x_r^{m_r} = \prod_{i=1}^{g}[a_i, b_i] \prod_{j=1}^{r} x_j \prod_{k=1}^{s} p_k = 1.$$

We then say Λ has signature (see [7])

$$(g, m_1, m_2, \dots\dots, m_r; s).$$

s is the number of parabolic classes, i.e., of conjugacy classes of maximal parabolic subgroup, and r is the number of elliptic classes. The m_i are the periods of Λ.

In Sec. 2, we will give a theorem concerning with the parabolic class number and then we will calculate parabolic class number of $K_0^m(n)$.

2. Main Theorems

Let Λ be a discrete subgroup of $PSL(2,\mathbb{R})$. When $x \in \mathbb{R} \cup \{\infty\}$ is a fixed point of a parabolic element of Λ, we say that x is a parabolic point of Λ. We also call a parabolic point of Λ a cusp of Λ.

We now give a theorem from [8], which is related to the cusps of Λ.

Theorem 1. *Let x be a cusp of Λ, and $\Lambda_x = \{T \in \Lambda : T(x) = x\}$. Then Λ_x is an infinite cyclic group. Moreover, any element of Λ_x is either identity or parabolic.*

Let us represent ∞ as $\frac{1}{0}\sqrt{m} = \frac{-1}{0}\sqrt{m}$ and let

$$\sqrt{m}\,\widehat{\mathbb{Q}} = \left\{ \frac{r}{s}\sqrt{m} : (r,s) = 1, r, s \in \mathbb{Z}, r \neq 0 \right\} \cup \{\infty\}.$$

Then

$$\sqrt{m}\,\widehat{\mathbb{Q}} = \left\{ \frac{r}{s}\sqrt{m} : (r,s) \in \mathbb{Z} \times \mathbb{Z},\ (r,s) = 1 \right\}$$

and any parabolic point of $H^m(\sqrt{m})$ is in $\sqrt{m}\,\widehat{\mathbb{Q}}$. Moreover, any point in $\sqrt{m}\,\widehat{\mathbb{Q}}$ is a parabolic point of $H^m(\sqrt{m})$. To see this, consider the following mapping

$$T(z) = \frac{(1 - rsm^2)z + r^2m^2\sqrt{m}}{-s^2m\sqrt{m}z + 1 + rsm^2}.$$

It is clear that T is a parabolic mapping and $\dfrac{r}{s}\sqrt{m}$ is a fixed point of T. Thus $\sqrt{m}\,\widehat{\mathbb{Q}}$ is the set of cusps of $H^m(\sqrt{m})$.

Let Γ be the modular group, and let $\Gamma_0(n)$ be the congruence subgroup of Γ such that $c \equiv 0(\mathrm{mod}\, n)$. Then, we give the following from [6].

Lemma 1. $\left|\Gamma : \Gamma_0(n)\right| = n \prod\limits_{p\,|\,n} \left(1 + \frac{1}{p}\right)$.

Lemma 2. *Let* $(m, n) = 1$*, then* $\left|H^m(\sqrt{m}) : K_0^m(n)\right| = n \prod\limits_{p\,|\,n} \left(1 + \frac{1}{p}\right)$. *If m is not prime to n, then* $\left|H^m(\sqrt{m}) : K_0^m(n)\right| = 2n \prod\limits_{p\,|\,n} \left(1 + \frac{1}{p}\right)$ *where p is prime to m.*

Proof. Let

$$H = \left\{T \in H(\sqrt{m}) \,:\, T(z) = \frac{az + bm\sqrt{m}}{cm\sqrt{m}z + d}\right\}$$

and let

$$H_0 = \{T \in H \,:\, c \equiv 0(\mathrm{mod}\, n)\}.$$

Then $H_0 \subset H$ and $H_0 \subset K_0^m(n)$. Moreover, it can be seen that H is conjugate to $\Gamma_0(m^3)$ and that H_0 is conjugate to $\Gamma_0(m^3n)$. This follows from the fact that $H = S^{-1}\Gamma_0(m^3)S$ and $H_0 = S^{-1}\Gamma_0(m^3n)S$ where $S(z) = \frac{1}{m\sqrt{m}}z$. In order to show that $H = S^{-1}\Gamma_0(m^3)S$ and $H_0 = S^{-1}\Gamma_0(m^3n)S$, let $T \in H$ and let

$$T(z) = \frac{az + bm\sqrt{m}}{cm\sqrt{m}z + d}.$$

Then

$$STS^{-1}(z) = ST(m\sqrt{m}z) = S((am\sqrt{m}z + bm\sqrt{m})/(cm\sqrt{m}m\sqrt{m}z + d))$$

and so

$$STS^{-1}(z) = S((m\sqrt{m}(az + b)/(cm^3z + d)) = (az + b)/(cm^3z + d).$$

This shows that $SHS^{-1} \subset \Gamma_0(m^3)$. Therefore, $SHS^{-1} = \Gamma_0(m^3)$, which implies that $H = S^{-1}\Gamma_0(m^3)S$. The fact that $H_0 = S^{-1}\Gamma_0(m^3n)S$ is proved similarly. It is clear that $\left|H^m(\sqrt{m}\,) : H\right| = \left|K_0^m(n) : H_0\right| = 2$. Thus we have

$$\left|H^m(\sqrt{m}) : K_0^m(n)\right| = \frac{|H^m(\sqrt{m}) : H_0|}{|K_0^m(n) : H_0|} = \frac{|H^m(\sqrt{m}\,) : H|\;|H : H_0|}{|K_0^m(n) : H_0|} = |H : H_0|.$$

Since $|H : H_0| = |S^{-1}\Gamma_0(m^3)S : S^{-1}\Gamma_0(m^3n)S| = |\Gamma_0(m^3) : \Gamma_0(m^3n)|$, we see that

$$\left|H^m(\sqrt{m}\,) : K_0^m(n)\right| = n \prod_{p\,|\,n} \Big(1 + \frac{1}{p}\Big).$$

If m is not prime to n, then the proof is similar. ■

Theorem 2. *Let Λ be a discrete subgroup of $PSL(2,\mathbb{R})$ and assume that Λ^* is a subgroup of finite index in Λ. Let P be the set of the parabolic points of Λ. Then P is the set of the parabolic points of Λ^* and the parabolic class number of Λ^* is the number of orbits of Λ^* on P.*

Proof. Let x be a point in P. Then $S(x) = x$ for some parabolic element of Λ. Since Λ^* is a subgroup of finite index in Λ, we see that $S^k \in \Lambda^*$ for some natural number k. Then S^k is a parabolic mapping and $S^k(x) = x$. Thus x is a cusp of Λ^*. By the above theorem, Λ^*_x is infinite cyclic group and any non-identity element of Λ^*_x is a parabolic element. Λ^*_x is maximal with respect to this property. For, if M is any cyclic subgroup such that $\Lambda^*_x \subset M$, then any element of M commutes every elements of Λ^*_x since M is commutative. Therefore every element of M fixes x. This shows that $M \subset \Lambda^*_x$, which implies that $M = \Lambda^*_x$. Thus Λ^*_x is a parabolic subgroup. Now let C be any parabolic subgroup of Λ^*. Then every element of C has a fixed point x and therefore $C \subset \Lambda^*_x$. This shows that $C = \Lambda^*_x$. In addition, x and y lie in the same orbit if and only if Λ^*_x and Λ^*_y are conjugate. Then the proof follows. ■

Theorem 3. *$H^m(\sqrt{m})$ is finitely generated.*

Proof. Since $\Gamma_0(m^3)$ is a subgroup of finite index in Γ, $\Gamma_0(m^3)$ is finitely generated and so H is finitely generated. In view of the fact that $|H^m(\sqrt{m}\,) : H| = 2$, it follows that $H^m(\sqrt{m})$ is finitely generated. ■

By virtue of the above theorems, it is seen that $K_0^m(n)$ is a finitely generated discrete subgroup of $H^m(\sqrt{m})$. Thus $K_0^m(n)$ is a Fuchsian group. Now we will give a lemma without proof, which we use later: The lemma appears in [1], page 73, as problem.

Lemma 3. *Let m and k be positive numbers. Then the number of the positive integers $\leq mk$ that are prime to m is $k\varphi(m)$.*

From now on, we sometimes represent $\dfrac{r}{s}\sqrt{m}$ as $r/s\sqrt{m}$ and we assume that m is a prime number.

Lemma 4. *Let $(m,n) = 1$ and let $r/s\sqrt{m} \in \sqrt{m}\,\widehat{\mathbb{Q}}$ with $m|s$. Then we can find some $T \in K_0^m(n)$ such that $T(r/s\sqrt{m}) = r_1/s_1\sqrt{m}$ with $(m, s_1) = 1$.*

Proof. Since $(m,n) = 1$, there exist some $a, b \in \mathbb{Z}$ such that $1 = m^3a - nb$. Let

$$T(z) = \frac{am\sqrt{m}z + b}{nz + m\sqrt{m}}.$$

Then $T \in K_0^m(n)$, and

$$T\Big(\frac{r}{s}\sqrt{m}\Big) = \frac{arm + bs}{(rn+s)\sqrt{m}} = \frac{ra + bs/m}{rn+s}\sqrt{m}.$$

It can be easily shown that $(m,\, rn + s) = 1$. If we take $r_1 = ar + bs/m$, and $s_1 = rn + s$, then $T(r/s\sqrt{m}) = r_1/s_1\sqrt{m}$ with $(m, s_1) = 1$.

Lemma 5. *Let $(k, s) = (m, s) = 1$. Then there exists some $T \in H^m(\sqrt{m})$ such that $T(k/s\sqrt{m}) = u\sqrt{m}$ where u is an integer.*

Proof. Since $(k, s) = (m, s) = 1$, we have $(s, km^2) = 1$. Therefore there exist two integers a and b such that $km^2a + sb = 1$. Since $(m^3a, b) = 1$, there exist two integers x and y such that $by - m^3ax = 1$. Let

$$T(z) = \frac{yz + xm\sqrt{m}}{am\sqrt{m}z + b}.$$

Then $T \in H^m(\sqrt{m})$ and

$$T(k/s\sqrt{m}) = \frac{u\sqrt{m}}{akm^2 + bs} = u\sqrt{m},$$

where $u = yk + mxs$. ∎

Lemma 6. *Let u and v be two integers. Then $u\sqrt{m}$ is conjugate to $v\sqrt{m}$ under $H^m(\sqrt{m})$ if and only if $u \equiv v (\operatorname{mod} m)$.*

Proof. Let $u \equiv v (\operatorname{mod} m)$. Then $u = v + rm$ for some integers r. Let $T(z) = z + rm\sqrt{m}$. It follows that $T(v\sqrt{m}) = u\sqrt{m}$. Conversely, assume that $u\sqrt{m}$ is conjugate to $v\sqrt{m}$ under $H^m(\sqrt{m})$. Then $T(u\sqrt{m}) = v\sqrt{m}$ for some $T \in H^m(\sqrt{m})$. It is easily seen that T is of the form

$$T(z) = \frac{az + bm\sqrt{m}}{cm\sqrt{m}z + d},\ ad - bcm^3 = 1.$$

Then

$$\frac{(au + bm)\sqrt{m}}{ucm^2 + d} = v\sqrt{m}.$$

Therefore $au + bm = (-1)^i v$, and $cum^2 + d = (-1)^i$ for some $i \in \{0, 1\}$. Then we see that $ad \equiv 1 (\operatorname{mod} m)$, $au \equiv (-1)^i v (\operatorname{mod} m)$, and $d \equiv (-1)^i (\operatorname{mod} m)$.Thus it follows that $u \equiv v (\operatorname{mod} m)$. ∎

Theorem 4. *The parabolic class number of $H^m(\sqrt{m})$ is m.*

Proof. It suffices to calculate the number of the orbits of $H^m(\sqrt{m})$ on $\sqrt{m}\ \widehat{\mathbb{Q}}$. Then from Lemmas 4, 5, and 6, the proof follows. ∎

Lemma 7. *Let $(m, n) = 1$, and $k/s\sqrt{m} \in \sqrt{m}\ \widehat{\mathbb{Q}}$ with $(k, s) = 1$. If $(m, s) = 1$, then there exists some $T \in K_0^m(n)$ such that $T(k/s\sqrt{m}) = k_1/s_1\sqrt{m}$ with $s_1 | n$.*

Proof. Since $(k, s) = (m, s) = 1$, we see that $(km^2, s) = 1$. Let $s_1 = (s, n)$. Then $s_1 = (s, n) = (s, km^2n)$. Therefore there exist some integers c_1, d_1 such that

$$km^2nc_1 + sd_1 = s_1.$$

Since $(d_1, km^2n/s_1) = 1$, there exists an integer k_0 such that $(d_1 - (km^2n/s_1)k_0, m^3n) = 1$. Let $d = d_1 - (km^2n/s_1)k_0$ and $c = c_1 + (s/s_1)k_0$. Then

$$km^2nc + sd = s_1.$$

On the other hand, $(d, cm^3n) = 1$, since $(d, m^3n) = (d, c) = 1$. Hence , we can find some integers x and y such that $xd - ycm^3n = 1$. If we take

$$T(z) = \frac{xz + ym\sqrt{m}}{cnm\sqrt{m}z + d},$$

then we have $T(k/s\sqrt{m}) = k_1/s_1\sqrt{m}$ with $s_1 | n$, where $k_1 = xk + ysm$ and $s_1 = cnm^2k + ds$. It is obvious that $T \in K_0^m(n)$. Moreover, it can be seen that $(k_1, s_1) = 1$. ∎

Lemma 8. *Let $(m, n) = 1$, $d_1 | n$, and $(a_1, d_1) = (a_2, d_1) = 1$. Then $a_1/d_1\sqrt{m}$ is conjugate to $a_2/d_1\sqrt{m}$ under $K_0^m(n)$ if and only if $a_1 \equiv a_2 (\mathrm{mod}\, tm)$ where $t = (d_1, n/d_1)$.*

Proof. Let $a_1 \equiv a_2 (\mathrm{mod}\, tm)$ and $n_1 = n/d_1$. Then $t = (d_1, n_1)$, and $(a_1a_2, d_1) = 1$. Furthermore, $(m, d_1) = 1$ since $(m, n) = 1$. Therefore, $(a_1a_2m, d_1) = 1$, and thus $(n_1a_1a_2m, d_1) = t$. This shows that $(n_1a_1a_2m^2, d_1m) = tm$. Since $tm | a_1 - a_2, m^2n_1a_1a_2x + d_1my = a_2 - a_1$ has a solution. That is, there exist two integers k and s such that $m^2n_1a_1a_2k + a_1 + d_1ms = a_2$. Hence, we obtain $a_1(1 + m^2n_1a_2k) + d_1ms = a_2$. If we take $a = 1 + m^2n_1a_2k$ and $b = s$, then we have $aa_1 + bmd_1 = a_2$. On the other hand, if we take $c = n_1d_1k$ and $d = 1 - m^2n_1a_1k$, then we obtain $m^2ca_1 + dd_1 = d_1$.

Furthermore,

$$\begin{aligned} ad - bcm^3 &= a(1 - m^2n_1a_1k) - bm^3n_1d_1k \\ &= a - (aa_1 + bmd_1)m^2n_1k = a - a_2m^2n_1k = 1. \end{aligned}$$

Let

$$T(z) = \frac{az + bm\sqrt{m}}{cm\sqrt{m}z + d}.$$

Then it is clear that $T \in K_0^m(n)$ and $T(a_1/d_1\sqrt{m}) = a_2/d_1\sqrt{m}$.

Now let $a_1/d_1\sqrt{m}$ be equivalent to $a_2/d_1\sqrt{m}$ by some $T \in K_0^m(n)$. Then it is easily seen that T is of the form

$$T(z) = \frac{az + bm\sqrt{m}}{cnm\sqrt{m}z + d} \quad \text{where} \quad ad - bcm^3n = 1.$$

Hence, we obtain

$$\frac{aa_1 + bmd_1}{cna_1m^2 + dd_1} = \frac{a_2}{d_1}.$$

Since

$$d(aa_1 + bmd_1) - bm(cna_1m^2 + dd_1) = a_1,$$

and

$$a(cna_1m^2 + dd_1) - cnm^2(aa_1 + bmd_1) = d_1,$$

we have $(aa_1 + bmd_1,\, cna_1m^2 + dd_1) = 1$. Therefore, there exists some $u = \mp 1$ such that

$$aa_1 + bmd_1 = ua_2,$$

and

$$cna_1m^2 + dd_1 = ud_1.$$

From the above equations and from the fact that $ad - bcm^3n = 1$, it follows that $aa_1 \equiv ua_2 (\operatorname{mod} tm)$, $d \equiv u (\operatorname{mod} tm)$, and $ad \equiv 1 (\operatorname{mod} mt)$. This shows that $a_1 \equiv u^2 a_2 (\operatorname{mod} tm)$, which implies that $a_1 \equiv a_2 (\operatorname{mod} tm)$. ∎

Lemma 9. *Let $(a_1, d_1) = (a_2, d_2) = 1$, $d_1 \mid n$, and $d_2 \mid n$. If $a_1/d_1\sqrt{m}$ is conjugate to $a_2/d_2\sqrt{m}$ under $K_0^m(n)$, then $d_1 = d_2$ where d_1 and d_2 are positive integers.*

Theorem 5. *Let m be a prime number and let n be prime to m. Then the parabolic class number of $K_0^m(n)$ is*

$$m \sum_{d \mid n} \varphi\Big(\Big(d,\, \frac{n}{d}\Big)\Big),$$

where φ is the Euler's function.

Proof. It suffices to calculate the number of orbits of $K_0^m(n)$ on $\sqrt{m}\ \widehat{\mathbb{Q}}$. By Lemmas 3 and 8, there exist $m\varphi((d,\, n/d))$ different orbits for each $d \mid n$. Then from Lemmas 4, 7, 8 and 9, it follows that the number of orbits of $K_0^m(n)$ on $\sqrt{m}\ \widehat{\mathbb{Q}}$ is

$$\sum_{d \mid n} m\varphi\Big(\Big(d,\, \frac{n}{d}\Big)\Big) = m \sum_{d \mid n} \varphi\Big(\Big(d,\, \frac{n}{d}\Big)\Big). \qquad \blacksquare$$

If we take $m = 1$, then $H(\sqrt{m})$ turn out to be the modular group Γ and $K_0^m(n)$ turn out to be the congruence subgroup $\Gamma_0(n)$ of the modular group Γ. Thus we can easily give the following.

Corollary 1. *The parabolic class number of $\Gamma_0(n)$ is*

$$\sum_{d \mid n} \varphi\Big(\Big(d,\, \frac{n}{d}\Big)\Big).$$

The following lemmas are the modifications of the above lemmas.

Lemma 10. *Let $(m, n) = 1$ and let $r/s\sqrt{m} \in \sqrt{m}\ \widehat{\mathbb{Q}}$ with $(r, s) = 1$ and $m|s$. Then we can find some $T \in H_0^m(n)$ such that $T(r/s\sqrt{m}) = r_1/s_1\sqrt{m}$ with $(m, s_1) = 1$.*

Lemma 11. *Let $(m, n) = 1$, and $k/s\sqrt{m} \in \sqrt{m}\ \widehat{\mathbb{Q}}$ with $(k, s) = 1$. If $(m, s) = 1$, then there exists some $T \in H_0^m(n)$ such that $T(k/s\sqrt{m}) = k_1/s_1\sqrt{m}$ with $s_1 | n$.*

Lemma 12. *Let $(m, n) = 1$, $d_1 \mid n$, and $(a_1, d_1) = (a_2, d_1) = 1$. Then $a_1/d_1\sqrt{m}$ is conjugate to $a_2/d_1\sqrt{m}$ under $H_0^m(n)$ if and only if $a_1 \equiv a_2 (\operatorname{mod} t)$ where $t = (d_1,\, n/d_1)$.*

Lemma 13. *Let $(a_1, d_1) = (a_2, d_2) = 1$, $d_1 \mid n$, and $d_2 \mid n$. If $a_1/d_1\sqrt{m}$ is conjugate to $a_2/d_2\sqrt{m}$ under $H_0^m(n)$, then $d_1 = d_2$ where d_1 and d_2 are positive integers.*

Now we can easily give the following.

Theorem 6. *Let m be a prime number and let n be prime to m. Then the parabolic class number of $H_0^m(n)$ is*

$$\sum_{d \mid n} \varphi\Big(\Big(d, \frac{n}{d}\Big)\Big).$$

Proof. It suffices to calculate the number of orbits of $H_0^m(n)$ on $\sqrt{m}\,\widehat{\mathbb{Q}}$. Then the proof follows from Lemmas 10, 11, 12, and 13. ∎

Corollary 2. *If m is not prime to n, then the parabolic class number of $K_0^m(n)$ is*

$$\sum_{d \mid nm^3} \varphi\Big(\Big(d, \frac{nm^3}{d}\Big)\Big).$$

Proof. If m is not prime to n, then $K_0^m(n) = H_0$ and therefore $K_0^m(n)$ is conjugate to $\Gamma_0(nm^3)$. The proof then follows. ∎

Now we generalize our results as follows. Let $q = m^r$ and let $H^q(\sqrt{m})$ be defined as follows

$$H^q(\sqrt{m}) = \big\{T \in H(\sqrt{m}) : a \equiv d \equiv 0 (\operatorname{mod} q) \text{ or } b \equiv c \equiv 0 (\operatorname{mod} q)\big\}.$$

Then $H^q(\sqrt{m})$ is finitely generated subgroup of $H(\sqrt{m})$. Furthermore, let $K_0^q(n) = H^q(\sqrt{m}) \cap H_0^m(n)$. We can show in a similar manner as above that $K_0^q(n)$ is of finite index in $H^q(\sqrt{m})$ and that the set of cusps of $H^q(\sqrt{m})$ is $\sqrt{m}\,\widehat{\mathbb{Q}}$. Thus the parabolic class number of $K_0^q(n)$ is the number of the orbits of $K_0^q(n)$ on $\sqrt{m}\,\widehat{\mathbb{Q}}$. Moreover, we have the following lemmas. The proof of those lemmas are the modifications of those of the above lemmas, Lemmas 4, 7, 8, and 9, respectively.

Lemma 14. *Let $(m, n) = 1$ and let $r/s\sqrt{m} \in \sqrt{m}\,\widehat{\mathbb{Q}}$ with $(r, s) = 1$ and $m|s$. Then we can find some $T \in K_0^q(n)$ such that $T(r/s\sqrt{m}) = r_1/s_1\sqrt{m}$ with $(m, s_1) = 1$.*

Lemma 15. *Let $(m, n) = 1$, and $k/s\sqrt{m} \in \sqrt{m}\,\widehat{\mathbb{Q}}$ with $(k, s) = 1$. If $(m, s) = 1$, then there exists some $T \in K_0^q(n)$ such that $T(k/s\sqrt{m}) = k_1/s_1\sqrt{m}$ with $s_1 \mid n$.*

Lemma 16. *Let $(m, n) = 1$, $d_1 \mid n$, and $(a_1, d_1) = (a_2, d_1) = 1$. Then $a_1/d_1\sqrt{m}$ is conjugate to $a_2/d_1\sqrt{m}$ under $K_0^q(n)$ if and only if $a_1 \equiv a_2 (\operatorname{mod} tq)$ where $t = (d_1, n/d_1)$.*

Lemma 17. *Let $(a_1, d_1) = (a_2, d_2) = 1$, $d_1 \mid n$, and $d_2 \mid n$. If $a_1/d_1\sqrt{m}$ is conjugate to $a_2/d_2 m\sqrt{m}$ under $K_0^q(n)$, then $d_1 = d_2$ where d_1 and d_2 are positive integers.*

Theorem 7. *The parabolic class number of* $K_0^q(n)$ *is*

$$q \sum_{d \mid n} \varphi\Big(\Big(d,\, \frac{n}{d}\Big)\Big)$$

in case m *is prime to* n *and the parabolic class number of* $H^q(\sqrt{m})$ *is* q.

Proof. It suffices to calculate the number of orbits of $K_0^q(n)$ on $\sqrt{m}\ \widehat{\mathbb{Q}}$. By Lemmas 14, 3, and 15, there exist $q\varphi((d, n/d)$ different orbits for each $d|n$. Then from Lemmas 15, 16, and 17, it follows that the number of orbits of $K_0^q(n)$ on $\sqrt{m}\ \widehat{\mathbb{Q}}$ is

$$\sum_{d \mid n} q\varphi\Big(\Big(d,\, \frac{n}{d}\Big)\Big) = q \sum_{d \mid n} \varphi\Big(\Big(d,\, \frac{n}{d}\Big)\Big).$$

■

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