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An Implicit Two-Phase Free Boundary Problem for the Heat Equation

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Abstract. The maximum principle and the fixed point argument are used for proving the existence and uniqueness of the solution of an implicit two-phase free boundary problem for the heat equation - denoted as "implicit" due to the absence of the derivative of the free boundary in the free boundary conditions.

1. Introduction

The free boundary problems for parabolic equation are classified in two classes: the class of problems in which the derivative of the free boundary appears explicitly in the free boundary conditions, as the Stefan problem and the class of problems in which the derivative of the free boundary is absent in the free boundary conditions,as the oxygen diffusion-consumption problem. The problems of the last class are called implicit free boundary problems.The Stefan like problems were studied by many authors, but there is not so much results on the implicit free boundary problems. A one-phase implicit free boundary problem for the heat equation was studied by one of this paper's authors [4]. In the two-phase Stefan problem with heat source along the free boundary, the standard equilibrium at the free boundary is expressed by:

$$u(s(t),t) = 0, \quad v(s(t),t) = 0,$$
$$u_x(s(t),t) - v_x(s(t),t) = -Ls'(t) + \lambda(s(t)),$$

where $u(x,t)$, $v(x,t)$ represent the temperature in the two phases, L is the latent heat. We will consider this problem, where $L = 0$, a two-phase implicit free boundary problem.

We will introduce some notations for formulating the problem. We denote by S_t for $t \in (0, \overline{T})$ the set of continuous on $[0, t]$ continuously differentiable on $(0, t]$ functions $s(\tau)$, satisfying $s(\tau) \geq S^0 > 0$, $|s'(\tau)| \leq S^1$ for $\tau \in (0, t)$, $s(0) = b, 0 < b < 1$, b, $\overline{T}$, S^0, S^1 being positive constants such that $\overline{T} < \min\left\{\frac{1-b}{S^1}, \frac{b}{S^1}\right\}, S^0 < b$. For $T \in (0, \overline{T}]$ we set

$$Q_T = \{(x, t) \,:\, 0 \leq x \leq 1, 0 \leq t \leq T\}$$

Problem (P): Find $(T, s(t), u(x,t), v(x,t))$ such that
1) $s(t) \in C^1[0, T]$, $0 < T \leq \overline{T}$, $s(0) = b$, $0 < b < 1$; the curve $x = s(t)$ divides the rectangular domain Q_T into two subdomains

$$D_T^{\pm} = \Big\{(x,t) \in Q_T \,:\, \pm\big(x - s(t)\big) \geq 0\Big\},$$

2) $u(x,t) \in C^{2,1}(D_T^-)$; $v(x,t) \in C^{2,1}(D_T^+)$,
3) the following equation and conditions are satisfied:

$$u_{xx} - u_t = 0, \quad \text{in} \quad D_T^-, \tag{1.1}$$

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq b, \tag{1.2}$$

$$u_x(0, t) = f(t), \quad 0 \leq t \leq T, \tag{1.3}$$

$$u(s(t), t) = 0, \quad 0 \leq t \leq T, \tag{1.4}$$

$$\gamma\, v_{xx} - v_t = 0, \quad \text{in} \quad D_T^+, \tag{1.5}$$

$$v(x, 0) = \phi(x), \quad b \leq x \leq 1, \tag{1.6}$$

$$v(1, t) = F(t), \quad 0 \leq t \leq T, \tag{1.7}$$

$$v(s(t), t) = 0, \quad 0 \leq t \leq T, \tag{1.8}$$

$$u_x(s(t), t) - v_x(s(t), t) = \lambda(s(t)), \tag{1.9}$$

where γ is a given positive constant, $\gamma \neq 1$, $\varphi(x)$, $f(t)$, $\phi(x)$, $F(t)$, $\lambda(x)$ are given functions satisfying the following conditions
$D_1)$ $\varphi(x) \in H_{3+\alpha}[0, b]$, $f(t) \in H_{1+\alpha}[0, \overline{T}]$, $\alpha \in (0, 1)$, $\varphi'(0) = f(0)$, $\varphi(b) = 0$.
$D_2)$ $\phi(x) \in H_{4+\alpha}[b, 1]$, $F(t) \in H_{2+\alpha}[0, \overline{T}]$, $\phi(1) = F(0)$, $\phi(b) = 0$.
$D_3)$ $\lambda(x) \in H_{2+\alpha}[0, 1]$, $|\phi'(b) + \lambda(b)| \neq 0$, $\overline{T} < \dfrac{|\phi'(b) + \lambda(b)|}{G_3}$.

Let

$$\mathcal{G} = \Big\{g(t) \in H_{1+\alpha}[0, \overline{T}) : 0 < G_2 \leq |g(t)| \leq G_1, \ |g'(t)| \leq G_3\Big\},$$

where $G_i, (i = 1, 2, 3)$ are the positive constants which will be suitably defined later.

2. The Existence and Uniqueness of the Solution

The proof is composed of following parts.
1) For every $g(t) \in \mathcal{G}$ there exists a unique solution of the Problem (P$'$) of finding the triple $(T, s(t), u(x,t))$ such that the equation (1.1), the conditions (1.2)–(1.4) and the condition

$$u_x(s(t), t) = g(t), \quad 0 \leq t \leq T$$

are satisfied.
2) For $s(t)$ satisfying the Problem (P′), there is a unique function $v(x,t) \in C_{2+\alpha}(D_T^+)$ satisfying the equation (1.5) and the conditions (1.6)–(1.8), (Problem (P″)).
3) The application $\widetilde{G} : g(t) \longmapsto \widetilde{g}(t)$, where $\widetilde{g}(t)$ is defined by

$$\widetilde{g}(t) = v_x(s(t),t) + \lambda(s(t)),$$

$s(t)$ solves the Problem (P′), $v(x,t)$ solves the Problem (1.5)–(1.8) is a contraction from $\mathcal{G}$ into itself. Then $(T,\ s(t),\ u(x,t),\ v(x,t))$ is the solution of Problem (P).

2.1. Some Auxiliary Problems

At first, we consider the following one-phase boundary problem.

Problem (P′): Find the triple $(T,\ s(t),\ u(x,t))$ such that
1) $s(t) \in C^1[0,T],\ 0 < T \leq \overline{T},\ s(0) = b,\ 0 < b < 1,$
2) $u(x,t) \in C^{2,1}(D_T^-),$
3) the following equation and conditions are satisfied

$$u_{xx} - u_t = 0, \quad \text{in} \quad D_T^-, \tag{2.1}$$

$$u(x,0) = \varphi(x), \quad 0 \leq x \leq b, \tag{2.2}$$

$$u_x(0,t) = f(t), \quad 0 \leq t \leq T, \tag{2.3}$$

$$u(s(t),t) = 0, \quad 0 \leq t \leq T, \tag{2.4}$$

$$u_x(s(t),t) = g(t), \quad 0 \leq t \leq T, \tag{2.5}$$

where $g(t)$ is a given function $\in \mathcal{G},\ \varphi'(b) = g(0)$.

Theorem 2.1. *Under the assumptions (D_1), (D_3) and $g(t) \in \mathcal{G}$ there exists a unique solution $(T, s(t), u(x,t))$ of the Problem (P'), $u(x,t) \in C_{2+\alpha}(D_T^-)$ and $s(t) \in \mathcal{S}_T$.*

Proof. The function

$$\widetilde{u}(x,t) = u_x(x,t) - g(t) \tag{2.6}$$

is the solution of the following problem

$$\widetilde{u}_{xx} - \widetilde{u}_t = g'(t) \quad \text{in} \quad D_T^-, \tag{2.7}$$

$$\widetilde{u}(x,0) = \widetilde{\varphi}(x), \quad 0 \leq x \leq b, \tag{2.8}$$

$$\widetilde{u}(0,t) = \widetilde{f}(t), \quad 0 \leq t \leq T, \tag{2.9}$$

$$\widetilde{u}(s(t),t) = 0, \quad 0 \leq t \leq T, \tag{2.10}$$

$$s'(t) = -\widetilde{u}_x(s(t),t)g^{-1}(t), \quad 0 \leq t \leq T, \tag{2.11}$$

where $\widetilde{\varphi}(x) = \varphi'(x) - g(0)$, $\widetilde{f}(t) = f(t) - g(t)$.

There is a unique solution $(T, s(t), \widetilde{u}(x,t))$ of the Problem (2.7)–(2.11) (see [1]). Hence $(T,\ s(t),\ u(x,t))$, where $u(x,t)$ is defined by (2.6), is the solution of the Problem (P'). This completes the proof. ∎

We set

$$\Sigma = \{b,\ \overline{T},\ \alpha,\ G_1,\ G_2,\ G_3,\ \|\varphi\|_{3+\alpha},\ \|f\|_{1+\alpha}\}.$$

Theorem 2.2. *Let (T_i, s_i, u_i) be the solutions of Problem (P') corresponding to $g_i(t) \in \mathcal{G}$, $i = 1, 2$ in the data (2.5). Under the assumptions (D_1), (D_3), there is a positive constant M_1 such that*

$$\big\|s_1 - s_2\big\|_{C^1[0,T]} \le M_1 T \big\|g_1 - g_2\big\|_{C^1[0,T]}, \tag{2.12}$$

where $T \le \min(T_1,\ T_2)$, M_1 depends on Σ, S^0, S^1.

Proof. Let

$$\delta(t) = s_1(t) - s_2(t), \quad u(x,t) = u_1(x,t) - u_2(x,t).$$

By the transformation

$$y = \frac{x}{s(t)}, \quad U_i(y,t) = u_{ix}(ys(t), t) \tag{2.13}$$

the function $U_i(y,t), (i = 1, 2)$ is the solution of the problem

$$s_i^{-2} U_{iyy} + y\ s_i'\ s_i^{-1} U_{iy} - U_{it} = g_i'(t) \ \text{ in } \ \mathcal{D}_T, \tag{2.14}$$

$$\begin{aligned} &U_i(y,0) = \widetilde{\varphi}(yb), \quad 0 \le y \le 1,\\ &U_i(0,t) = \widetilde{f}(t), \quad 0 \le t \le T,\\ &U_i(1,t) = 0, \quad 0 \le t \le T, \end{aligned} \tag{2.15}$$

$$s_i'(t) = -U_{iy}(1,t) s_i^{-1}(t) g_i^{-1}(t), \quad 0 \le t \le T, \tag{2.16}$$

where $\mathcal{D}_T = \{(x,t) : 0 \le x \le 1,\ 0 \le t \le T\}$.

The function $U(y,t) = U_1(y,t) - U_2(y,t)$ is the solution of the problem

$$\begin{aligned} \mathcal{L}U = s_1^{-2} U_{yy} - U_t = -y\ U_{2y}\Big[\frac{\delta'(t)}{s_1} - \delta(t)\ s_2' \frac{1}{s_1 s_2}\Big] - y \frac{s_1'}{s_1} U_y \\ + \frac{(s_1 + s_2)\delta(t)}{s_1^2 s_2^2} U_{2yy} + \big(g_1'(t) - g_2'(t)\big) \equiv Q(y,t) \quad \text{in} \quad \mathcal{D}_T, \end{aligned} \tag{2.17}$$

$$U(y,0) = U(0,t) = U(1,t) = 0. \tag{2.18}$$

Let

$$\Delta g' = \max_{0 \le t \le T} |g_1'(t) - g_2'(t)|,$$

$$\Delta g = \max_{0 \le t \le T} |g_1(t) - g_2(t)|,$$

we have

$$\Delta g \le T \Delta g'. \tag{2.19}$$

Following Fasano and Primicerio [2], we have

$$\begin{aligned} \max_{0 \le y \le 1} |U_y(y,t)| \le M_2 \Big\{ \int_0^t (t-\tau)^{-\frac{1}{2}} \max_{0 \le y \le 1} |U_y(y,\tau)| d\tau \\ + t^{\frac{1}{2}} \|\delta'\|_t + t^{\frac{1}{2}} \Delta g' \Big\}, \quad 0 \le t \le T. \end{aligned} \tag{2.20}$$

By using Gronwall lemma we get

$$\max_{0\le y\le 1} |U_y(y,t)| \le M_3\Big\{t\|\delta'\|_t + t\Delta g'\Big\}, \quad 0\le t\le T, \tag{2.21}$$

and

$$|\delta'(t)| \le M_3M_4T(\|\delta'\|_t + \Delta g') + M_5 t\|\delta'\|_t + M_6T\Delta g',$$

with $\Delta g \le T\Delta g'$, $0\le t\le T$, positive constants $M_i, i = 2,3,4,5,6$ depend on Σ, S^0, S^1.

Then with T sufficiently small, there is a positive constant

$$M_1 = \frac{(M_3M_4 + M_6)}{1-(M_3M_4+M_5)T}$$

such that

$$\big\|\delta\big\|_{C^1[0,T]} \le M_1T\big\|\Delta g\big\|_{C^1[0,T]},$$

M_1 decreases when T decreases. Thus the theorem is proved. ∎

We denote $\sigma = \{\gamma,\ \|\phi\|_{4+\alpha},\ \|F\|_{2+\alpha}\}$.

For $s(t)$ satisfying the Problem P', we consider the following boundary problem with fixed boundary.

Problem (P″): Find the function $v(x,t)\in C^{2,1}(D_T^+)$ satisfying

$$\begin{aligned}
&\gamma v_{xx} - v_t = 0, \quad \text{on} \quad D_T^+,\\
&v(x,0) = \phi(x), \quad b\le x\le 1,\\
&v(1,t) = F(t), \quad 0\le t\le T,\\
&v(s(t),t) = 0, \quad 0\le t\le T.
\end{aligned}$$

It is well known that there exists a unique solution $v(x,t)\in C_{2+\alpha}(D_T^+)$ of the Problem (P″).

2.2. The Existence and Uniqueness of Solution of Problem (P)

Theorem 2.3. *There exists a unique solution* $(T, s(t),\ u(x,t), v(x,t))$ *of the Problem* (P).

Indeed for $s(t)$ satisfying the Problem(P') corresponding to $g(t)\in\mathcal{G}$ in the condition (2.5) and for $v(x,t)$ satisfying the Problem (P"), we set

$$\widetilde{g}(t) = v_x(s(t),t) + \lambda(s(t)). \tag{2.22}$$

Thus the application $\widetilde{G} : g(t) \longmapsto \widetilde{g}(t)$ is defined. We will show that $\widetilde{G}$ is a contraction from a closed subset of the Banach space $C^1[0,\overline{T}]$ into itself.

Lemma 2.1. *The solution* $v(x,t)$ *of Problem* (P'') *belongs to* $C[D_T^+]$. *Moreover we have the following estimation*

$$|v_{xt}(x,t)| \le M_8, \tag{2.23}$$

where the positive constant M_8 *depends on* Σ, σ, S^0, S^1.

Proof. The function

$$z(x,t) = v_t(x,t), \tag{2.24}$$

satisfies the problem

$$\begin{aligned} &\gamma z_{xx} - z_t = 0, \ \text{ in } \ D_T^+, && (2.25)\\ &z(x,0) = \phi\text{"}(x), \quad b \le x \le 1, \\ &z(0,t) = F'(t), \quad 0 \le t \le T, && (2.26)\\ &z(s(t),t) = -v_x(s(t),t)s'(t), \quad 0 \le t \le T. \end{aligned}$$

By setting

$$y = \frac{x-1}{s(t)-1}, \tag{2.27}$$

$$Z(y,t) = z\big(y(s(t)-1)+1,t\big), \tag{2.28}$$

$$\phi_1(y) = \phi\text{"}\big(y(b-1)+1\big),$$

the function $Z(y,t)$ satisfies the equation

$$\gamma[s-1]^{-2}Z_{yy} + ys'(s-1)^{-1}Z_y - Z_t = 0, \quad \text{in} \quad \mathcal{D}_T,$$

and the condition

$$\begin{aligned} &Z(y,0) = \phi_1(y), \quad 0 \le y \le 1, \\ &Z(0,t) = F'(t), \quad 0 \le t \le T, \\ &Z(1,t) = V_y(1,t)\frac{s'(t)}{1-s(t)}, \quad 0 \le t \le T, \end{aligned}$$

where $V(y,t) = v\big(y(s(t)-1)+1,t\big)$.

We have

$$|Z_y(y,t)| \le M_8, \tag{2.29}$$

where the positive constant M_8 depends on Σ, σ, S^0, S^1 (see [5, p. 502]). ■

From (2.24),(2.28),(2.29) we get (2.23).

Lemma 2.2. *The application $\widetilde{G}$ carries $\mathcal{G}$ into itself.*

Proof. From (2.22) and Lemma 2.1, it follows that

$$|\widetilde{g}'(t)| \le G_3$$

where G_3 depends on b, $\overline{T}$, S^1, $\|\lambda\|_{H_{2+\alpha}}$. Hence we obtain

$$\begin{aligned} |\widetilde{g}(t)| &= |v_x(s(t),t) + \lambda(s(t))| \le |v_x(s(t),t)| + |\lambda(s(t))| \\ &= \frac{|V_y(1,t)|}{|1-s(t)|} + |\lambda(s(t))| \le \frac{M_9}{1-(b+S^1\overline{T})} + \|\lambda\|_{H_{2+\alpha}}, \\ |\widetilde{g}(t)| &= |\widetilde{g}(0) - \int_0^t \widetilde{g}'(t)dt| \ge |\widetilde{g}(0)| - |\int_0^t \widetilde{g}'(t)dt| \\ &\ge |v_x(b,0) + \lambda(b)| - G_3 t \ge |\phi'(b) + \lambda(b)| - G_3\overline{T}. \end{aligned}$$

We take

$$G_1 = \frac{M_9}{(1-b) - S^1\overline{T}} + \|\lambda\|_{H_{2+\alpha}} > 0 \quad \text{since} \quad \overline{T} < \frac{1-b}{S^1},$$
$$G_2 = |\phi'(b) + \lambda(b)| - G_3\overline{T} > 0 \quad \text{since} \quad \overline{T} < \frac{|\phi'(b) + \lambda(b)|}{G_3},$$

where the positive constant M_9 depends on Σ, σ, S^0, S^1. Then $\widetilde{g}(t) \in \mathcal{G}$.

Lemma 2.3. *Let $s_1(t), s_2(t)$ be the solutions of Problem (P') corresponding to $g_1, g_2 \in \mathcal{G}$ in the data (2.5) and let $v_1(x,t)$ and $v_2(x,t)$ be the solutions of Problem $(P")$ corresponding to $s_1(t), s_2(t)$. We have the following estimation*

$$\big|v_{1x}(s_1(t),t) - v_{2x}(s_2(t),t)\big| \le M_{11}\big\|s_1 - s_2\big\|_{C^1[0,T]}, \quad 0 \le t \le T, \tag{2.30}$$

$$\big|v_{1xx}(s_1(t),t) - v_{2xx}(s_2(t),t)\big| \le M_{12}\big\|s_1 - s_2\big\|_{C^1[0,T]}, \tag{2.31}$$

$$\big|v_{1xt}(s_1(t),t) - v_{2xt}(s_2(t),t)\big| \le M_{13}\big\|s_1 - s_2\big\|_{C^1[0,T]}, \tag{2.32}$$

for $t \in [0,T]$, positive constants M_{11}, M_{12}, M_{13} depend on Σ, σ, S^0, S^1.

Proof. We denote $V_i(y,t) = v\big(y(s_i(t)-1)+1,t\big)$, $(i=1,2)$ the solution of the problem

$$\begin{aligned}
\gamma(s_i-1)^{-2}V_{iyy} + ys_i'(s_i-1)^{-1}V_{iy} - V_{it} = 0 \text{ in } \mathcal{D}_T,\\
V_i(y,0) = \phi_1(y), \quad b \le y \le 1,\\
V_i(0,t) = F(t), \quad 0 \le t \le T,\\
V_i(1,t) = 0, \quad 0 \le t \le T.
\end{aligned}$$

Then $\overline{V}(y,t) = V_1(y,t) - V_2(y,t)$ solves the problem

$$\begin{aligned}
\gamma(s_1-1)^{-2}\overline{V}_{yy} + ys_1'(s_1-1)^{-1}\overline{V}_y - \overline{V}_t = \frac{\gamma(s_1+s_2+2)\delta}{(s_1-1)^2(s_2-1)^2}V_{2yy}\\
-y\Big(\frac{\delta'}{s_1-1} - s_2\frac{\delta}{(s_1-1)(s_2-1)}\Big)V_{2y} = h(y,t),\\
\overline{V}(y,0) = \overline{V}(0,t) = \overline{V}(1,t) = 0,
\end{aligned}$$

in $\mathcal{D}_T$, where

$$\begin{aligned}
|h(y,t)| &\le \gamma\frac{|s_1+s_2+2|}{(s_1-1)^2(s_2-1)^2}|\delta||V_{2yy}| + |y|\frac{|\delta'|}{s_1-1}|V_{2y}|\\
&\quad + y||s_2|\frac{|\delta|}{(s_1-1)(s_2-1)}|V_{2y}|\\
&\le M_{10}\big\|\delta\big\|_{C_1[0,T]},
\end{aligned}$$

M_{10} depends on Σ, σ, S^0, S^1. We get

$$\|V_{1y}(1,t) - V_{2y}(1,t)\|_{C^1[0,T]} \le M_{11}\|s_1 - s_2\|_{C^1[0,T]}. \tag{2.33}$$

(see [3, p. 238]) and hence (2.30) is proved. The estimation (2.31), (2.32) follow from the Lemmas 2.1 and 2.3.

Proof of Theorem 2.3. From the definition of $\widetilde{g}(t)$ and Lemma 2.3, we get

$$\begin{aligned} |\widetilde{g}_1'(t) - \widetilde{g}_2'(t)| &\le |v_{1xx}(s_1(t),t)s_1'(t) + v_{1xt}(s_1(t),t) + \lambda_x(s_1(t))s_1'(t) - \\ &\quad - v_{2xx}(s_2(t),t)s_2'(t) - v_{2xt}(s_2(t),t) - \lambda_x(s_2(t))s_2'(t)| \\ &\le |v_{1xx}(s_1(t),t) - v_{2xx}(s_2(t),t)||(s_1'(t)| \\ &\quad + |v_{2xx}(s_2(t),t)||s_1'(t) - s_2'(t)| \\ &\quad + |v_{1xt}(s_1(t),t) - v_{2xt}(s_2(t),t)| \\ &\quad + |\lambda_x(s_1(t)) - \lambda_x(s_2(t))||s_1'(t)| \\ &\quad + |\lambda_x(s_2(t))||s_1'(t) - s_2'(t)| \\ &\le M_{14}\|\delta\|_{C^1[0,T]}, \quad 0 \le t \le T, \end{aligned} \tag{2.34}$$

M_{14} depends on Σ, σ, S^0, S^1, $\|\lambda\|_{H_{2+\alpha}}$. Therefore

$$|\widetilde{g}_1(t) - \widetilde{g}_2(t)| = |\int_0^t (\widetilde{g}_1'(t) - \widetilde{g}_2'(t))dt| \le M_{14}T\|\delta\|_{C^1[0,T]}, \quad 0 \le t \le T. \tag{2.35}$$

From (2.34),(2.35) we get

$$\|\widetilde{g}_1 - \widetilde{g}_2\|_{C_1[0,T]} \le M_{15}\|s_1 - s_2\|_{C^1[0,T]}. \tag{2.36}$$

where the positive constant M_{15} depends on Σ, σ, S^0, S^1, $\|\lambda\|_{H_{2+\alpha}}$.

Combining (2.12),(2.36), we obtain

$$\|\widetilde{g}_1 - \widetilde{g}_2\|_{C^1[0,T]} \le M_1M_{15}T\|g_1 - g_2\|_{C^1[0,T]}.$$

For T sufficiently small, $M_1M_{15}T < 1$. Then the application $\widetilde{G}$ is a contraction from $\mathcal{G}$ into itself. From Banach principle $\widetilde{G}$ has a unique fixed point. Then $T, s(t), u(x,t)$ solves the Problem (P'), $v(x,t)$ solves the corresponding (P'') and the condition (1.9) is satisfied. This completes the proof of Theorem 2.3. ■

References

1. A. Fasano and M. Primicerio, Cauchy type free boundary problems for nonlinear parabolic equations, *Riv. Mat. Univ. Parma* **5** (1979) 615–634.
2. A. Fasano and M.Primicerio, Free boundary problems for nonlinear parabolic equations with nonlinear free boundary conditions, *J. Math. Anal. Appl.* **72** (1979) 247–273.
3. A. Friedman, *Partial Differential Equation of Parabolic Type,* Translation in Russian, Mir, Moscow, 1968.
4. Nguyen Dinh Tri, On a free boundary problem for the heat equation, Free boundary problems: theory and applications, *Research Notes in Math.,* Vol II, Pitman, London, 1983, 397–406.
5. O. A. Ladyzenskaja, V. A. Solonikov, and N. N. Ural'ceva, *Linear and Quasi-linear Equation of Parabolic Type,* Nauka, Moscow, 1967 (Russian).