# M1 Algebraic Topology, exercices

**1.** The cone $CX$ of a pointed space $(X, x_0)$ is defined as the quotient of $X \times [0,1]$ by the equivalence relation which shrinks subsets $(x, 1)$ $(\forall x \in X)$ and $(x_0, t)$, $0 \le t \le 1$ to a single point. This point will be considered to be the base point of the cone.
rove that the identity map of a cone is (pointed) homototpic to the constant map to the base point. (Such a space is called contractible).

**2.** The suspension $\Sigma X$ of a pointed space $(X, x_0)$ is defined as the quotient of $X \times [0,1]$ by the equivalence relation which shrinks subsets $(x, 1)$, $(x, 0)$, $(\forall x \in X)$ and $(x_0, t)$, $0 \le t \le 1$ to a single point - the base point of $\Sigma X$. Prove thate $\Sigma S^{n-1}$ is homeomorphic to $S^n$.
*Hint:* We can begin by proving similar claim on the unreduced suspension - the quotient of $X \times [0,1]$ by the relation which shrinks the subset $(x, 0)$, $\forall x \in X$ to a single point and the subset $(x, 0)$, $\forall x \in X$ to another point. Then use the exercice 15.

**3.** Let $S^\infty$ denote the direct limit space of the direct system of spheres $S^n$ *via* inclusions

$$S^n \hookrightarrow S^{n+1} \quad (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_{n+1}, 0)$$

Prove that a map from $S^\infty$ to an arbitrary space $X$ is continous if and only if its restriction on each $S^n$ is continous.
Prove that a compact subset of $S^\infty$ is always contained in a certain $S^n$.

**4.** Prove that $S^\infty$ is contractible.
*Hint:* We should first prove that the identity map of $S^\infty$ is (freely) homotopic to the map induced by the *décalage* map, i.e. the restriction to $S^\infty$ of the linear map defined by $s(e_i) = e_{i+1}$ on the canonical base. Then use the exercise 15.

**5.1.** Let consider $\mathrm{GL}(n, \mathbb{C})$ and $U(n)$ as subspaces of $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$. Prove that these spaces are connected and that $\mathrm{GL}(n, \mathbb{C})$ is open and densed in $M_n(\mathbb{C})$. Prove that $U(n)$ is compact.

**5.2.** Prove that $\mathrm{GL}(n, \mathbb{C})$ is homeomorphic to $U(n) \times \mathbb{R}^{n^2}$.

**6.** Prove that the group $U(n)$ is homeomorphic to $S^1 \times SU(n)$.

**7.** Formulate analogous results for groups $\mathrm{GL}(n, \mathbb{R})$ and $O(n)$ and prove them.

**8.** Prove that the group $SU(2)$ is homeomorphic to $S^3$.

**9.** Using quaternions to prove that the group $SO(3)$ is the quotient space of sphere $S^3$ by identification of antopodal points.
Do the same thing for the group $SO(4)$ and $S^3 \times S^3$.

**10.** Let consider the canonical action of the group of complex numbers having module 1 on the sphere $S^3 \subset \mathbb{C}^2$. Prove that the orbit space (quotient space) is homeomorphic to sphere $S^2$.

**11.** Prove that $TS^2 \subset S^2 \times \mathbb{R}^3$

$$\{(v, w) | v \in S^2 \;\; w \in R^3, \;\; |w| = 1, \;\; \text{et} \;\; < v | w >= 0\}$$

is homeomorphic to $SO(3)$, where $< v|w >$ denotes the usual scalar product.

**12.** Let $G$ be a topological group and $\pi_1(G, e)$ its fundamental group. Prove that on the set $\pi_1(G, e)$ we can introduce another multiplication of two loops $\gamma(t)\lambda(t)$ by using the multiplication of the group $G$. Prove that this product is coincide with the usual product of the fundamental group. As corollary, we have that the group $\pi_1(G, e)$ is abelian.

**13.** Prove the same claims as in the exercise 12, repalcing the group $G$ by a $H$-space $X$.

**14.** Let the real infinite projective space $\mathbb{RP}^\infty$ be the direct limit of $\mathbb{RP}^n$ *via* inclusions $\mathbb{RP}^n \to \mathbb{RP}^{n+1}$

$$(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_{n+1}, 0).$$

Prove that $\mathbb{RP}^\infty$ is a $H$-space. Prove the same thing for the complex infinite projective space $\mathbb{CP}^\infty$. Prove that they are abelian topological groups.

**15.** Let $\mathbb{B}^n$ be the closed unit ball in $\mathbb{R}^n$ and $C$ a simple $C^1$-path (image of the closed interval$[0, 1]$ under a $C^1$-differential injective map) in $\mathbb{B}^n$. Prove that the quotient space $\mathbb{B}^n/C$ is homeomorphic to $\mathbb{B}^n$. Prove the same result for $\mathbb{R}^n$, $S^n$ or any differentiable manifold in place of $\mathbb{B}^n$.

**16.** Prove that the space $\mathbb{S}^1 \times \mathbb{S}^1$ with one point punctured is homotopic to the wedge product of two circles $\mathbb{S}^1$.

**17.** Given a pointed space $(X, x_0)$. Let $\Omega(X)$ denote the space of closed paths based at $x_0$, equipped with the compact-open topology.
Prove that if $X$ is a metric space the above topology is coincide with that defined by uniform convergence.
Prove that $\Omega(X)$ is a $H$-space.
Prove that

$$\pi_0(\Omega(X)) \cong \pi_1(X, x_0)$$

**18.** Prove that any continous map from $S^1$ to itself having no fixed point is (freely) homotopic to the identity map.

**19.** Let $P$ be a polynomial with complex coefficients having no roots on $S^1$. Prove that the number of roots of $P$ (comptées avec multiplicité) with module strictement inférieur à 1 is the degree of the map from $S^1$ to $S^1$ given by $z \mapsto \frac{P(z)}{|P(z)|}$.

**20.** Prove that $\mathbb{R}^2$ et $\mathbb{R}^n$, $n > 2$, can not be homeomorphic to each other.